Minimizing Uplink Cellular Outage Probability in Interference Limited Rayleigh and Nakagami Wireless Fading Channels

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Abstract: We propose a game theoretic non-cooperative algorithm to optimize the induced outage probability in an uplink cellular interference limited wireless Rayleigh and Nakagami fading channels. We achieve this target by maximizing the certainty equivalent margin (CEM). We derive a closed-form formula of the outage probability in Nakagami flat-fading channels, then we show that minimizing the induced outage fading probability for both Rayleigh and Nakagami channels is equivalent to maximizing CEM. We present a non-cooperative power control algorithm using the game theory framework. Through this non-cooperative game, we argue that the best decision in such an environment is for all users to transmit at the minimum power in their corresponding strategy profiles. This finding considerably simplifies the implementation of the proposed game.

Keywords: cellular network, code-division-multiple-access (CDMA), outage probability, power control, non-cooperative game (NPG).

1 Introduction

Game theory was first brought to the literature by John Von Neumann and Oskar Morgenstern in 1944 [7], and by the end of 1970's game theory became a framework of great value in the researcher's resources whenever there is a situation in which an agent's action depends on the other agents' actions. The basic aim of game theory is the method where deliberate dealings between intelligent agents, produces outcomes according to the agents' desires [2], [10]. In a non-cooperative game, an agent counters the actions of other agents by selecting a strategy from his strategy space in aiming to optimize a utility function that quantifies its quality of service level (QoS).

In an uplink cellular data communications systems, cellular users (agents) prefer to have a high signal-to-interference ratio (SIR) at the base station (BS) while sending at the lowest possible power, this helps to have longer battery life and helps cope with interference problem in CDMA systems [8]. Also, it is important to have a high SIR, as this means having a low error rate, a more reliable system, and high channel capacity, which results in higher attainable bit rates [14], [9].

In game theoretic power control algorithms for cellular systems, each cellular user maximizes his utility function by transmitting at a feasible power level to reciprocate the other cellular users' aggregate transmitted powers. This action generates a series of power vectors that converges to an operating point that is called Nash equilibrium (NE) point. However, due to nature of the non-cooperative algorithms that lack cooperation between the users, it is not guaranteed that this operating point will be efficient, in the sense that it may not be the most preferred operating point by all cellular users [11].

Game theoretic framework was first exploited in [13] to provide solutions and insights for the power control problem for CDMA communication systems, then was more exploited in [11]

and [12]. The research work in [5] presented a bound that relates the induced Rayleigh channel fading outage probability to the SIR margin. Where SIR margin is just the SIR with the signal power and the interference noise power are replaced by their statistical means. Then they used Perron-Frobenius eigenvalue theory to allocate the cellular users' transmit powers in a manner that minimizes the system outage probability. For more feasible case, that is, when the transmitted powers are constrained along with other constraints, they modeled the optimization problem as a geometric programming problem to find a global solution. However, the global solution was based on a centralized algorithm that run on the BS to collect data about all path gains of the links between transmitters and their corresponding receivers in the cellular communications system.

In this paper the problem addressed in [5] to minimize the outage probability and the certainty-equivalent margin for Rayleigh wireless fading channels, is solved in a non-centralized manner under both Rayleigh and Nakagami fading channels. We also provide a simpler proof of the strong relationship between the two problems, namely: Minimizing the fading induced outage probability, and maximizing the CEM. We then present a closed-form formula of the outage probability in a Nakagami flat-fading channel. Using the outage probability formula, we prove that both problems mentioned above remain equivalent in Nakagami flat fading channels.

What is left from the paper is presented as follows: The system model we studied in this paper is described in section 2. Deriving a closed-form formula of the induced outage probability in an interference limited Nakagami flat fading channels is performed in Section 3. The strong correlation between the minimization of the outage probability and the maximization of the CEM is presented in 4. In section 5, our proposed Non-cooperative power control game (NPG) is discussed. Simulation results are presented and explained in section 6, and our conclusions are casted in section 7.

2   System model

The system under investigation in this paper is similar to that studied in [3]. In [3], the solutions for optimizing the system CEM and the system uplink outage probability under Rayleigh flat-fading channels were introduced. In this paper however, we extend our study to figure out a non-centralized power control game theoretic-algorithm for also Nakagami flat-fading channels. Thus, the system under study is as follows: Assume we have one cell with N cellular users communicating with BS in the uplink direction. Which can be seen as N transmitter-receiver pairs, where the ith transmitter transmits messages at a power level p_i from his action profile P_i to the corresponding ith receiver (the BS). The power profile P_i set is assumed to be convex set. The received power level at receiver i from transmitter k is given by:

$$H_{i,k}\chi_{i,k}p_k \tag{1}$$

in which $H_{i,k} > 0$ is the link gain between transmitter k to receiver i. $H_{i,k}$ captures the effect of spreading gain and/or cross correlation between codes in CDMA (code division multiple access) systems. $\chi_{i,k}, \; i,k = 1, 2, ..., N$ are exponentially independent distributed (id) random variables with mean equal to 1 in a wireless Rayleigh flat-fading channel. On the other hand in a Nakagami flat-fading channel, $\chi_{i,k}, \; i,k = 1, 2, ..., N$ are Gamma id random variables with mean also equal to 1. Hence, the power received from user k at receiver i is modeled as Exponential (Gamma) random variable with expected value expressed as:

$$\mathrm{E}\{H_{i,k}\chi_{i,k}p_k\} = H_{i,k}p_k \tag{2}$$

In an interference limited fading channels, the background additive white Gaussian noise (AWGN) effect is considered minor compared to the power interference from the other active users. Therefore, the signal-to-interference ratio of the ith user at the corresponding receiver is given by:

$$SIR_i = \frac{H_{i,i}\chi_{i,i}p_i}{\sum_{k\neq i}^{N} H_{i,k}\chi_{i,k}p_k}. \tag{3}$$

It is clear from (3) that SIR_i is a random variable, which is a ratio of an exponentially (Gamma) distributed random variable to a summation of independent exponentially (Gamma) distributed random variables with different means under Rayleigh (Nakagami) fading link. The outage probability O_i, defined as the probability that the SIR of an active user i will be less than a threshold SIR_{th}, then for user i:

$$
\begin{aligned}
O_i &= \Pr\{SIR_i \leq SIR_{th}\} \\
&= \Pr\{H_{i,i}\chi_{i,i}p_i \leq SIR_{th} \sum_{k\neq i}^{N} H_{i,k}\chi_{i,k}p_k\}
\end{aligned}
\tag{4}
$$

This probability was derived in [5] for a Rayleigh flat fading channel:

$$O_i = 1 - \prod_{k\neq i}^{N} \frac{1}{1 + SIR_{th}\frac{H_{i,k}p_k}{H_{i,i}p_i}} \tag{5}$$

The system outage probability, O is described as:

$$O = \max_{1\leq i\leq N} O_i \tag{6}$$

Where O may be considered as on of the performance metrics of the cellular system and the adopted power control algorithm. The CEM of user i is given as the ratio of his certainty-equivalent SIR to the corresponding threshold SIR. Mathematically,

$$CEM_i = \frac{SIR_i^{ce}}{SIR_{th}} \tag{7}$$

where, the certainty-equivalent SIR (SIR_i^{ce}) of user i is described as the ratio of the mean of his received power at the corresponding receiver to the mean of the interference from the active other users in the system, that is,

$$SIR_i^{ce} = \frac{H_{i,i}p_i}{\sum_{k\neq i}^{N} H_{i,k}p_k} \tag{8}$$

While, the system certainty-equivalent SIR, SIR^{ce} is given as:

$$SIR^{ce} = \min_{1\leq i\leq N} SIR_i^{ce} \tag{9}$$

Thus, the system CEM is given by:

$$CEM = \min_{1\leq i\leq N} CEM_i \tag{10}$$

The CEM is yet another cellular system performance metric for the deployed power control algorithm.

3   Interference limited outage probability under Nakagami flat fading Channel

In this section we present our extension to the previous work in [3]. Here, we derive a closed-form formula of the outage probability in an interference limited cellular communications system under Nakagami flat fading channels. Having said that, let $\alpha_{i,k}$ be the fading coefficient of the link connecting transmitter k with receiver i, then $\alpha_{i,k}$ has a Nakagami distribution:

$$f^{\alpha_{i,k}}(\omega) = \frac{2\,m^m}{\Gamma(m)\,\Omega^m}\,\omega^{2m-1}\,\exp\left(-\frac{m}{\Omega}\omega^2\right). \tag{11}$$

where Ω is defined as [9]

$$\Omega = E\{\alpha_{i,k}^2\}$$

It controls the spread of Nakagami distribution. The fading figure denoted by m is given as [9]

$$m = \frac{\Omega^2}{E\{(\alpha_{i,k}^2 - \Omega)^2\}} \ge \frac{1}{2},$$

it stands as a measure of the harshness of the fading channel, for example when $m = \infty$, it corresponds to a nonfading channel, and if $m = 1/2$ it corresponds to the Half-normal which is the most severe fading channel. The coming lemma presents a closed-form of the outage probability in a flat-fading Nakagami channels.

Lemma 1. *Under Nakagami flat fading channel with $\Omega = 1$ and $m = 2$, the outage probability O_i is formulated as:*

$$O_i \;\;=\;\; 1 - \left(N - \sum_{k \neq i}^{N}\left[\frac{1}{1 + \frac{SIR_{th}\,H_{i,k}\,p_k}{H_{i,i}\,p_i}}\right]\right)\prod_{k \neq i}^{N}\left[\frac{1}{1 + \frac{SIR_{th}\,H_{i,k}\,p_k}{H_{i,i}\,p_i}}\right]^2 \tag{12}$$

Proof: For simplicity, define $Y_{i,k} \triangleq H_{i,k}\chi_{i,k}p_k$, where $\chi_{i,k} = \alpha_{i,k}^2$, and hence $Y_{i,k}$ is a Gamma distributed random variable, that is:

$$f^{Y_{i,k}}(\omega) = \frac{\xi_{i,k}^m}{\Gamma(m)}\,\omega^{m-1}\,\exp(-\xi_{i,k}\,\omega) \tag{13}$$

Where $\xi_{i,k} = \frac{m}{\Omega\,H_{i,k}\,p_k}$. In the following calculations we set $m = 2$ and $\Omega = 1$. Henceforth, the probability O_i of user i can be expressed as:

$$\begin{aligned}
O_i \;\;&=\;\; \Pr\{Y_{i,i} \le SIR_{th}\,\sum_{k \neq i}^{N}Y_{i,k}\} \\
&=\;\; \int_0^{X_{i,k}} f^{Y_{i,i}}(\omega)d\omega
\end{aligned} \tag{14}$$

Where $X_{i,k} \triangleq SIR_{th}\sum_{k \neq i}^{N}Y_{i,k}$. It is clear that $X_{i,k}$ is a summation of independent Gamma distributed random variables with different means. For the sake of simplicity, we first find a conditional outage probability in (14) $O_i^c(x_{i,k}) = \Pr(O_i|X_{i,k} = x_{i,k})$, with $x_{i,k}$ is a realization of the random variable $X_{i,k}$. Therefore, we have the following:

$$\begin{aligned}
O_i^c(x_{i,k}) \;\;&=\;\; \xi_{i,i}^2\int_0^{x_{i,k}}\omega\,\exp(-\xi_{i,i}\,\omega)d\omega \\
&=\;\; 1 - (1 + \xi_{i,i}\,x_{i,k})\,\exp(-\xi_{i,i}\,x_{i,k})
\end{aligned} \tag{15}$$

And hence, outage probability O_i can be found as follows:

$$
\begin{aligned}
O_i &= \int_0^\infty O_i^c(\omega) \; f^{X_{i,k}}(\omega) d\omega \\
&= \mathrm{E}\left\{O_i^c(X_{i,k})\right\} \\
&= \mathrm{E}\left\{1 - (1 + \xi_{i,i} \; X_{i,k}) \; \exp(-\xi_{i,i} X_{i,k})\right\}
\end{aligned}
\tag{16}
$$

Using the following intermediate equations:

$$
\int_0^\infty \omega^n \; \exp(-a\,\omega) d\omega = \frac{\Gamma(n+1)}{a^{n+1}},
\tag{17}
$$

$$
\mathrm{E}\left\{\exp(-\xi_{i,i} \, SIR_{th} \, Y_{i,k})\right\} = \left[\frac{1}{1 + \frac{SIR_{th} \, H_{i,k} \, p_k}{H_{i,i} \, p_i}}\right]^2,
\tag{18}
$$

and

$$
\mathrm{E}\{Y_{i,k} \; \exp(-\xi_{i,i} \, SIR_{th} \, Y_{i,k})\} = H_{i,k} \, p_k \left[\frac{1}{1 + \frac{SIR_{th} \, H_{i,k} \, p_k}{H_{i,i} \, p_i}}\right]^3
\tag{19}
$$

we end up with the following formula of the outage probability O_i,

$$
\begin{aligned}
O_i &= 1 - \prod_{k \neq i}^N \left[\frac{1}{1 + \frac{SIR_{th} \, H_{i,k} \, p_k}{H_{i,i} \, p_i}}\right]^2 - \sum_{k \neq i}^N \frac{SIR_{th} \, H_{i,k} \, p_k}{H_{i,i} \, p_i} \left[\frac{1}{1 + \frac{SIR_{th} \, H_{i,k} \, p_k}{H_{i,i} \, p_i}}\right]^3 \\
&\quad \times \prod_{l \neq i, l \neq k}^N \left[\frac{1}{1 + \frac{SIR_{th} \, H_{i,l} \, p_l}{H_{i,i} \, p_i}}\right]^2
\end{aligned}
\tag{20}
$$

To make (20) look like (12), we just need to substitute the following equations in (20):

$$
\prod_{l \neq i, l \neq k}^N \left[\frac{1}{1 + \frac{SIR_{th} \, H_{i,l} \, p_l}{H_{i,i} \, p_i}}\right]^2 = \left(1 + \frac{SIR_{th} \, H_{i,k} \, p_k}{H_{i,i} \, p_i}\right)^2 \prod_{l \neq i}^N \left[\frac{1}{1 + \frac{SIR_{th} \, H_{i,l} \, p_l}{H_{i,i} \, p_i}}\right]^2
\tag{21}
$$

and

$$
\begin{aligned}
\sum_{k \neq i}^N \frac{SIR_{th} \, H_{i,k} \, p_k}{H_{i,i} \, p_i} \left[\frac{1}{1 + \frac{SIR_{th} \, H_{i,k} \, p_k}{H_{i,i} \, p_i}}\right] &= \sum_{k \neq i}^N \left(1 - \left[\frac{1}{1 + \frac{SIR_{th} \, H_{i,k} \, p_k}{H_{i,i} \, p_i}}\right]\right) \\
&= \left(N - 1 - \sum_{k \neq i}^N \left[\frac{1}{1 + \frac{SIR_{th} \, H_{i,k} \, p_k}{H_{i,i} \, p_i}}\right]\right)
\end{aligned}
\tag{22}
$$

which concludes the proof. $\qquad\square$

4   Relation between outage probability and CEM

4.1   Rayleigh flat fading channel

We present the following proposition which was first introduced as a remark in [5], and then we deliver our own simpler argument.

Theorem 2. *Minimization of the fading induced outage probability O_i and maximization of CEM_i of user i in a Rayleigh fading channel are equivalent in terms of power allocation.*

Proof: Here, we need to keep in mind that the problem under investigation is to minimize the system outage probability using non-centralized algorithm, that is, every cellular user allocates his transmit power level such that his outage probability is minimized. Mathematically speaking,

$$\min_{p_i \in P_i} \; O_i \; = \; \min_{p_i \in P_i} \; 1 - \prod_{k \neq i}^{N} \frac{1}{1 + SIR_{th} \frac{H_{i,k}p_k}{H_{i,i}p_i}} \tag{23}$$

and this of course is equivalent to the following:

$$\max_{p_i \in P_i} \; \prod_{k \neq i}^{N} \frac{1}{1 + SIR_{th} \frac{H_{i,k}p_k}{H_{i,i}p_i}} \tag{24}$$

or

$$\min_{p_i \in P_i} \; \prod_{k \neq i}^{N} \left(1 + SIR_{th} \frac{H_{i,k}p_k}{H_{i,i}p_i} \right) \tag{25}$$

Using the fact that the Logarithmic function is a monotonic function, therefore minimizing (25) is identical to minimizing the following:

$$\min_{p_i \in P_i} \; \sum_{k \neq i}^{N} \log \left(1 + SIR_{th} \frac{H_{i,k}p_k}{H_{i,i}p_i} \right) \tag{26}$$

Using the logarithmic inequality $\log(x) \leq x - 1$, an upper bound of (26) can be given as:

$$\min_{p_i \in P_i} \quad \sum_{k \neq i}^{N} \; \log \left(1 + SIR_{th} \frac{H_{i,k}p_k}{H_{i,i}p_i} \right)$$
$$\leq \quad \min_{p_i \in P_i} \; \sum_{k \neq i}^{N} SIR_{th} \frac{H_{i,k}p_k}{H_{i,i}p_i} \tag{27}$$

It is intuitive to notice that the right-hand side of the inequality in (27) is equivalent to maximizing the CEM, that is

$$\max_{p_i \in P_i} \; \frac{H_{i,i}p_i}{SIR_{th} \sum_{k \neq i}^{N} H_{i,k}p_k} \; = \; \max_{p_i \in P_i} \; CEM_i \tag{28}$$

$\square$

4.2 Nakagami flat fading channel

Similar to Rayleigh flat fading channels we have the following proposition for Nakagami flat fading channels.

Theorem 3. *Minimization of the fading induced outage probability O_i and maximization of CEM_i of user i under a Nakagami fading channel are equivalent in terms of power allocation.*

Proof: For simplicity, let us define $T_{i,k} \triangleq SIR_{th} \frac{H_{i,k} p_k}{H_{i,i} p_i}$. Minimizing the outage probability O_i given in (12) is equivalent to the following problem:

$$\max_{p_i \in P_i} \left\{ \left(N - \sum_{k \neq i}^{N} \left[\frac{1}{1 + T_{i,k}} \right] \right) \prod_{k \neq i}^{N} \left[\frac{1}{1 + T_{i,k}} \right]^2 \right\} \tag{29}$$

or

$$\max_{p_i \in P_i} \left\{ \ln \left(N - \sum_{k \neq i}^{N} \left[\frac{1}{1 + T_{i,k}} \right] \right) + 2 \sum_{k \neq i}^{N} \ln \left(\frac{1}{1 + T_{i,k}} \right) \right\}$$

$$= \max_{p_i \in P_i} \left\{ \ln \left(1 + \sum_{k \neq i}^{N} \left(1 - \frac{1}{1 + T_{i,k}} \right) \right) + 2 \sum_{k \neq i}^{N} \ln \left(1 - \frac{T_{i,k}}{1 + T_{i,k}} \right) \right\}$$

$$\leq \max_{p_i \in P_i} \left\{ \sum_{k \neq i}^{N} (1 - \frac{1}{1 + T_{i,k}}) - 2 \sum_{k \neq i}^{N} \frac{T_{i,k}}{1 + T_{i,k}} \right\}$$

$$= \max_{p_i \in P_i} \left\{ \sum_{k \neq i}^{N} - \frac{T_{i,k}}{1 + T_{i,k}} \right\} \tag{30}$$

Where we used the inequality $\ln(x) \leq x - 1$ again and the assumption that $T_{i,k} << 1$ (SIR is high) to get the inequality in (30). Now one can see that the problem in (30) is equivalent to:

$$\max_{p_i \in P_i} \left\{ \sum_{k \neq i}^{N} \frac{1 + T_{i,k}}{T_{i,k}} \right\} = \max_{p_i \in P_i} \left\{ N - 1 + \sum_{k \neq i}^{N} \frac{H_{i,i} p_i}{SIR_{th} T_{i,k} p_k} \right\} \tag{31}$$

and since $N - 1$ is a constant common for all users, therefore (31) is equivalent to:

$$\min_{p_i \in P_i} \left\{ \frac{\sum_{k \neq i}^{N} SIR_{th} H_{i,k} p_k}{H_{i,i} p_i} \right\} \tag{32}$$

Finally, this problem is clearly equivalent to the problem of maximizing CEM_i. $\qquad \square$

In the next section we present a simple non-cooperative game $G2$, and we show that $G2$ results in an optimal power allocation to maximize the system CEM, minimize the system outage probability O, and minimize the total transmitted power.

5 Power control algorithm to optimize the outage probability

In this section we introduce a non-cooperative power control game as defined in (34), in which user i attempts to find the optimal transmit power level p_i from his strategy space P_i

that enables him to obtain a maximum possible certainty-equivalent-margin CEM^*, instead of maximizing CEM_i directly as given below:

$$G1 \quad : \quad \max_{p_i \in P_i} \ CEM_i,$$

$$= \max_{p_i \in P_i} \ \frac{H_{i,i}p_i}{SIR_{th} \ \sum_{k \neq i}^{N} H_{i,k}p_k}, \forall \, i = 1, 2, ..., N \tag{33}$$

The reason for avoiding maximizing CEM_i directly is that the game $G1$ in (33) has an objective function CEM_i which is linear in p_i given the power vector p_{-i} of all users except for the ith user. This will lead user i to send at the maximum power in his strategy space, as will all users. This selfish act will result in all users having very small CEMs. Due to this reason we propose the following non-cooperative power control game $G2$ defined as follows:

$$G2: \quad \max_{p_i \in P_i} \ CEM_i^*(n), \ n = 1, 2, \cdots$$
$$\text{subject to}$$

$$p_i(n) = \min \left(p_{i-max}, \frac{CEM_i^*(n) \ SIR_{th} \ \sum_{k \neq i}^{N} H_{i,k}p_k}{H_{i,i}} \right) \tag{34}$$

Seemingly, game $G2$ is a multistage game where in the nth stage, user i transmits at a power level $p_i(n)$ that enables him to attain a constant $CEM_i^*(n)$ (e.g. $CEM_i^*(n) = 1$), then if transmit power level $p_i(n)$ is feasible, i.e., $p_i(n) < p_{i-max}$, user i seeks a higher value of CEM_i^* at the $(n+1)$th stage of the game. Mathematically speaking, user i sets $CEM_i^*(n+1) > CEM_i^*(n)$ and finds $p_i(n+1)$ such that $CEM_i^*(n+1) = \frac{H_{i,i}p_i(n+1)}{SIR_{th} \ \sum_{k \neq i}^{N} H_{i,k}p_k}$ and so forth until he/she is satisfied with the value of $CEM_i^*(n)$. However, it turns out that game $G2$ is a one shot game, that is, it has a Nash equilibrium point with all users able to attain their maximum CEM_i^* in the first stage ($n = 1$) as we show in the following lemma. The lemma also guarantees the existence, uniqueness, and optimality of the Nash equilibrium point:

Lemma 4. *The non-cooperative game $G2$ with strategy space $P_i = [p_{i-min}, \, p_{i-max}]$ for user i and with $p_{i-min} > 0$, has a unique Nash equilibrium operating point. Also, this Nash equilibrium point is optimal in the sense that it corresponds to the minimum total transmitted power of all users. Users in game $G2$ will attain their maximum possible CEM_i^* in the first trial.*

Proof: At each time instance, user i updates his transmit power p_i in order to satisfy the following equation:

$$CEM_i^* = \frac{H_{i,i}p_i}{SIR_{th} \ \sum_{k \neq i}^{N} H_{i,k}p_k}, \tag{35}$$

for a target CEM_i^*. Exploiting the linearity of equation (35), we rewrite it in a matrix form as follows:

$$\text{A p} = \text{p} , \tag{36}$$

where the entries of the matrix A are given by:

$$A(i,j) = \begin{cases} \frac{SIR_{th} \, CEM_i^* \, H_{i,j}}{H_{i,i}}, & \text{if } i \neq j \\ 0, & \text{if } i = j \end{cases}$$

And $p = (p_1, p_2, ..., p_N)$ is the vector of transmit powers of all users. Since $H_{i,j} > 0$, matrix A is a nonnegative irreducible matrix. By the Perron-Frobenius theorem, the largest eigenvalue in magnitude of matrix A is real and positive [6], [1], and [4]. Therefore, if it happens that 1 is

this eigenvalue of A, then the solution p will be the eigenvector that corresponds to eigenvalue 1. Otherwise, the only solution of equation (36) is the trivial solution p = 0. Since p = 0 is not a feasible solution, each user will transmit at the lowest power level in his strategy space. Therefore, the Nash equilibrium point of game $G2$ is unique and corresponds to an optimal point that minimizes the total transmitted power of all users.

Equation (36) holds true for any $CEM_i^* \geq 1$ including the maximum value that users can attain at the equilibrium point. This implies that the cellular users will attain their maximum possible CEM_i^* in their first trial, i.e., $G2$ is a one-shot game. $\square$

It is easy to notice that if user i increases his power unilaterally to improve hisher CEM_i and O_i, at least one other user in the network will be harmed.

Lemma 4 implies that in an interference-limited wireless Rayleigh and Nakagamai flat fading channels the best policy is for all users to transmit at the minimum power in their corresponding strategy spaces. The question that may arise is: How do we guarantee the quality of service (QoS), e.g. SIR at the BS? The following lemma answers this question:

Lemma 5. *If users in an interference-limited wireless fading channels are not able to attain a satisfactory QoS (SIR) by transmitting at their minimum power vector $p_{min} = (p_{1-min}, p_{2-min}, ..., p_{N-min})$ in their corresponding strategy spaces, they will not be able to attain a satisfactory QoS by transmitting at any other power vector $p^l = (p_1^l, p_2^l, ..., p_N^l)$ larger than p_{min}, $p^l > p_{min}$ component wise.*

Proof: Let $\mathcal{N} = (1, 2, ..., N)$ be the indexing set of all active users in the cell. Suppose CEM_i^{min} is the CEM value user i attained by assuming that all users are transmitting at the minimum powers in their corresponding strategy spaces, that is, $\forall i \in \mathcal{N}$:

$$CEM_i^{min} = \frac{H_{i,i}\, p_{i-min}}{SIR_{th} \sum_{k \neq i}^{N} H_{i,k}\, p_{k-min}}, \tag{37}$$

and suppose that CEM_i^l is the value of CEM the ith user attains assuming all users transmitting at p^l, henceforth, $\forall i \in \mathcal{N}$ we have:

$$CEM_i^l = \frac{H_{i,i}\, p_i^l}{SIR_{th} \sum_{k \neq i}^{N} H_{i,k}\, p_k^l} \tag{38}$$

To prove lemma 5, it is enough to prove that there is a subset of users $\mathcal{K}_N \subset \mathcal{N}$, such that:

$$CEM_n^l \leq CEM_n^{min}, \forall n \in \mathcal{K}_N \tag{39}$$

Suppose this is not true, therefore

$$CEM_i^l > CEM_i^{min}, \forall i \in \mathcal{N} \tag{40}$$

Define $d_i^{l,min}$ as:

$$
\begin{aligned}
d_i^{l,min} &= CEM_i^l - CEM_i^{min} \\
&= \frac{1}{SIR_{th}} \left[\frac{H_{i,i}\, p_i^l}{\sum_{k \neq i}^{N} H_{i,k}\, p_k^l} - \frac{H_{i,i}\, p_{i-min}}{\sum_{k \neq i}^{N} H_{i,k}\, p_{k-min}} \right], \forall i \in \mathcal{N}
\end{aligned} \tag{41}
$$

Without loss of generality, $\forall i \in \mathcal{N}$ let $p_i^l = \delta_i\, p_{i-min}$, where $\delta_i > 1$. Then, (41) can be written as:

$$d_i^{l,min} = \frac{H_{i,i}\, p_{i-min}}{SIR_{th}} \left[\frac{\delta_i}{\sum_{k \neq i}^{N} \delta_k\, H_{i,k}\, p_{k-min}} - \frac{1}{\sum_{k \neq i}^{N} H_{i,k}\, p_{k-min}} \right], \forall i \in \mathcal{N} \tag{42}$$

Table 1: Equilibrium values of CEM_i and O_i for the first 10 users in a Rayleigh flat fading channel using Perron-Frobenius theorem and our NPG game $G2$ at $SIR_{th} = 3$

	Results using Perron-Frobenius theorem [5]			Results of game $G2$		
i	p_i	CEM_i	O_i	p_i	CEM_i	O_i
1	0.1292	13.7107	0.0703	0.0100	14.9809	0.0645
2	0.1162	13.7107	0.0703	0.0100	16.6629	0.0582
3	0.1476	13.7107	0.0703	0.0100	13.0747	0.0736
4	0.1482	13.7107	0.0703	0.0100	12.9549	0.0742
5	0.1290	13.7107	0.0703	0.0100	14.9275	0.0647
6	0.1192	13.7107	0.0703	0.0100	16.3274	0.0594
7	0.1297	13.7107	0.0703	0.0100	15.0442	0.0643
8	0.1312	13.7107	0.0703	0.0100	14.6970	0.0657
9	0.1327	13.7107	0.0703	0.0100	14.4091	0.0670
10	0.1361	13.7107	0.0703	0.0100	14.2906	0.0675
average		13.7107	0.0703	average	13.7823	0.0704

The inequality in (40) implies that $d_i^{l,min} > 0$ for all $i \in \mathcal{N}$, therefore

$$\delta_i \sum_{k \neq i}^{N} H_{i,k}\, p_{k-min} > \sum_{k \neq i}^{N} \delta_k\, H_{i,k}\, p_{k-min}, \forall\, i \in \mathcal{N} \tag{43}$$

For simplicity define $x_{i,k} = H_{i,k}\, p_{k-min}$, then (43) can be expressed as:

$$\delta_i \sum_{k \neq i}^{N} x_{i,k} > \sum_{k \neq i}^{N} \delta_k\, x_{i,k}, \forall\, i \in \mathcal{N} \tag{44}$$

If we set $i = 1$ in the above equation we get the following result:

$$\delta_1 > \delta_m, m = 2, 3, ..., N, \tag{45}$$

while if we set $i = 2$, we get the following:

$$\delta_2 > \delta_m, m = 1, 3, ..., N, \tag{46}$$

From equation (45), we have $\delta_1 > \delta_2$, while from (46) we find $\delta_2 > \delta_1$, so we have a contradiction. This proves the statement in equation (39), and this concludes the proof of lemma 5. $\square$

6 Simulation results

In our simulations, we considered one cell with $N = 50$ receiver and transmitter pairs in the uplink mode. The path gains $H_{i,k}$ were generated according to a uniform distribution on the interval $[0, 0.001]$ for all $i \neq k \in \mathcal{N}$ and $H_{i,i} = 1\ \forall\, i \in \mathcal{N}$. Game $G2$ was run for different values of the threshold signal-to-interference ratios (SIR_{th}) in the interval $[3, 10]$.

In Fig.1 we depict the system certainty-equivalent margin values (CEM) resulting from game G2 versus the threshold signal-to-interference ratios, SIR_{th} under both Rayleigh and Nakagami flat fading channels. While in Fig.2, we present the resulting system outage probability, O

Table 2: Equilibrium values of CEM_i and O_i for the first 10 users in a Rayleigh flat fading channel using Perron-Frobenius theorem and our NPG game $G2$ at $SIR_{th} = 10$

i	p_i	CEM_i	O_i	p_i	CEM_i	O_i
	Results using Perron-Frobenius theorem [5]			Results of game $G2$		
1	0.1439	4.0985	0.2159	0.0100	4.0243	0.2194
2	0.1541	4.0985	0.2159	0.0100	3.7254	0.2347
3	0.1266	4.0985	0.2159	0.0100	4.5414	0.1971
4	0.1497	4.0985	0.2159	0.0100	3.8603	0.2275
5	0.1375	4.0985	0.2159	0.0100	4.1930	0.2116
6	0.1378	4.0985	0.2159	0.0100	4.1877	0.2118
7	0.1096	4.0985	0.2159	0.0100	5.3148	0.1711
8	0.1514	4.0985	0.2159	0.0100	3.8264	0.2293
9	0.1398	4.0985	0.2159	0.0100	4.1414	0.2139
10	0.1384	4.0985	0.2159	0.0100	4.1889	0.2118
average		4.0985	0.2159	average	4.1255	0.2156

Table 3: Equilibrium values of CEM_i and O_i for the first 10 users in a Nakagami flat fading channel using Perron-Frobenius theorem and our NPG game $G2$ at $SIR_{th} = 3$

i	p_i	CEM_i	O_i	p_i	CEM_i	O_i
	Results using Perron-Frobenius theorem			Results of game $G2$		
1	0.1368	13.7415	0.0580	0.0100	14.2272	0.0561
2	0.1501	13.7415	0.0580	0.0100	12.9034	0.0618
3	0.1305	13.7415	0.0580	0.0100	14.6840	0.0543
4	0.1360	13.7415	0.0580	0.0100	14.2485	0.0560
5	0.1270	13.7415	0.0580	0.0100	15.0958	0.0528
6	0.1539	13.7415	0.0580	0.0100	12.5887	0.0633
7	0.1499	13.7415	0.0580	0.0100	12.8494	0.0620
8	0.1402	13.7415	0.0580	0.0100	13.7151	0.0581
9	0.1577	13.7415	0.0580	0.0100	12.2655	0.0649
10	0.1552	13.7415	0.0580	0.0100	12.5932	0.0633
average		13.7415	0.0580	average	13.8343	0.0581
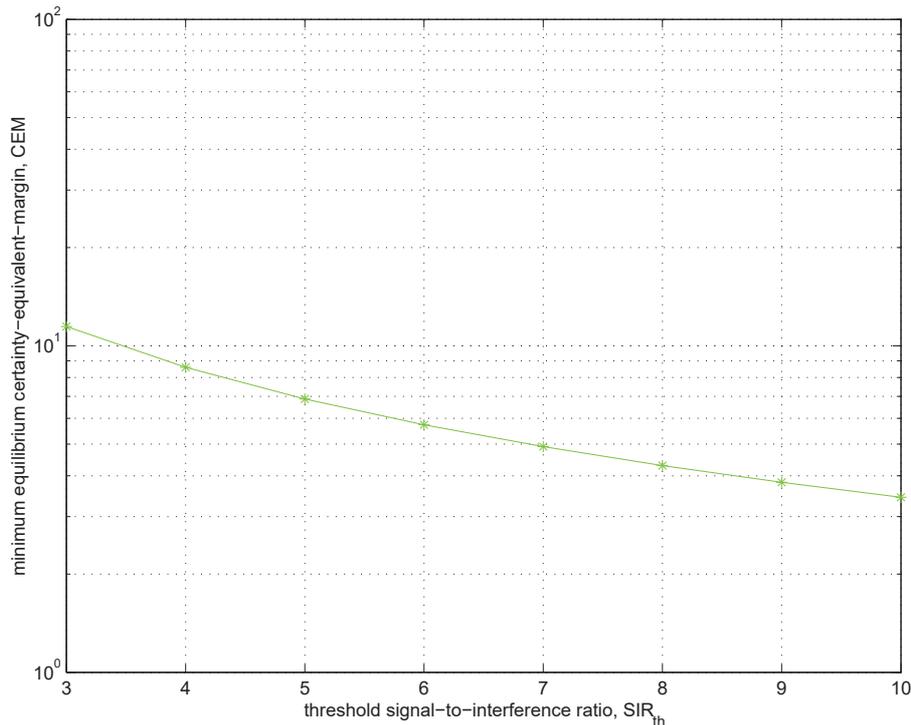
Figure 1: Minimum equilibrium CEM in Rayleigh and Nakagami fading channels versus the threshold SIR ratio.

of a Rayleigh fading channel($*$) and a Nakagami fading channel ($\circ$) versus the threshold SIR (SIR_{th}) compared to the minimum bound $1/(1 + CEM)$ (solid line) and the upper bound $1 - e^{-1/CEM}$ (dashed line) derived in [5]. In this figure, as one can see, in the Rayleigh case, the upper bound overlaps with the equilibrium outage probability which is the output of game $G2$. The results shown in Fig.1 and Fig.2 happened to be very close to the results obtained using Perron-Frobenius theorem [5] (in Rayleigh fading channel) but at a lower power allocation. This is clear in tables 2 - 4. In these tables we are casting CEM_i and O_i for the first 10 users as evaluated by the Perron-Frobenius theorem in [5] and as equilibrium outcomes of the NPG game $G2$ of both channel models: Rayleigh flat fading channel and Nakagami flat fading channel. The averages of CEM_i and O_i presented in the tables are calculated for all the 50 users in the system. We noticed that the average value of CEM obtained through NPG game $G2$ was higher than that obtained by the Perron-Frobenius theorem based algorithm for all values of SIR_{th}. The average value of O achieved through NPG game $G2$ was sometimes lower and sometimes higher than that achieved by Perron-Frobenius theorem. By examining tables 2 - 4, one can see that by transmitting at Perron-Frobenius power eigenvector many users attain lower values of CEM than they obtained when all users transmitted at the minimum power vector. This exactly agrees with what was proved in lemma 5. Finally, notice that in Fig.2 results show that users can achieve better performance in a Nakagami fading channel with ($m = 2$) than in a Rayleigh channel. This is expected since a Nakagami fading channel with fading figure $m = 2$ represents a less severe fading channel than a Rayleigh fading channel which is a Nakagami fading channel with fading figure $m = 1$.
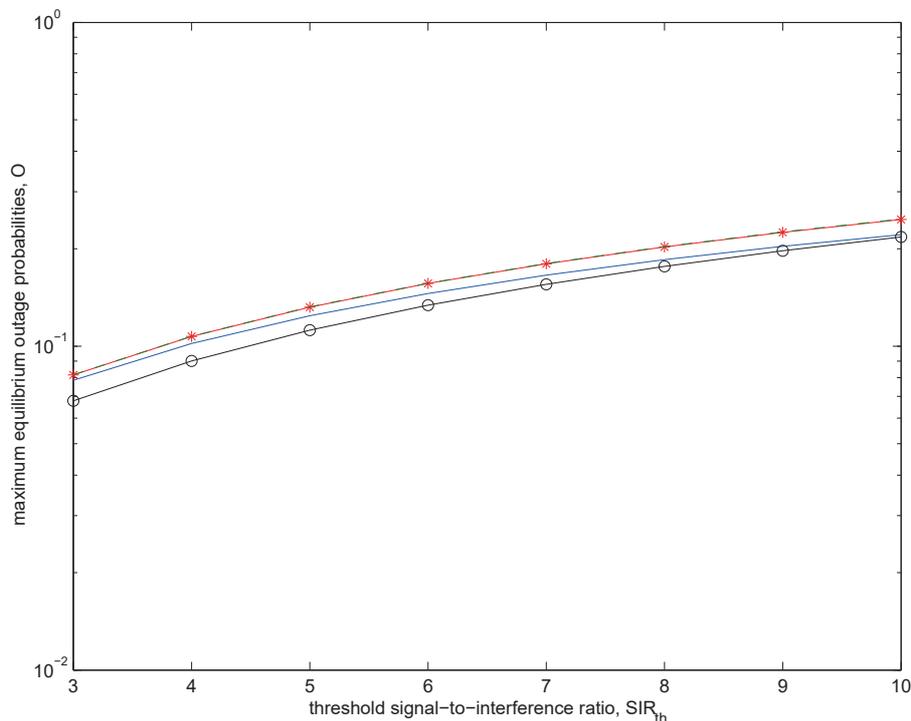
Figure 2: Maximum equilibrium Rayleigh fading induced outage probability ($*$), the lower bound of the outage probability $\frac{1}{1+CEM}$ (solid line), the upper bound $1 - e^{-1/CEM}$ (dashed line) and the maximum outage probability in a Nakagami channel ($\circ$) versus the threshold SIR ratio.

Table 4: Equilibrium values of CEM_i and O_i for the first 10 users in a Nakagami flat fading channel using Perron-Frobenius theorem and our NPG game $G2$ at $SIR_{th} = 10$

Results using Perron-Frobenius theorem			Results of game $G2$		
i	p_i	CEM_i	O_i	p_i	CEM_i	O_i
1	0.1539	4.0590	0.1906	0.0100	3.7011	0.2078
2	0.1432	4.0590	0.1906	0.0100	3.9928	0.1936
3	0.1384	4.0590	0.1906	0.0100	4.1278	0.1876
4	0.1553	4.0590	0.1906	0.0100	3.6718	0.2094
5	0.1378	4.0590	0.1906	0.0100	4.1369	0.1872
6	0.1305	4.0590	0.1906	0.0100	4.3997	0.1766
7	0.1593	4.0590	0.1906	0.0100	3.5804	0.2143
8	0.1553	4.0590	0.1906	0.0100	3.6847	0.2087
9	0.1456	4.0590	0.1906	0.0100	3.9305	0.1965
10	0.1483	4.0590	0.1906	0.0100	3.8569	0.2000
average		4.0590	0.1906	average	4.0948	0.1903

Note: header row shows i, p_i, CEM_i, O_i then p_i, CEM_i, O_i.

7 Conclusion

We proved the tight relationship between the two optimization problems: minimizing the system outage probability and maximizing the system CEM under Rayleigh and Nakagami flat fading wireless channels and. A closed-form of the outage probability under Nakagami flat fading channel was also provided. We then proposed an asynchronous distributed non-cooperative power control game-theoretic algorithm to optimize the system CEM and the system outage probability. Using the proposed non-cooperative game $G2$, we proved that the best power allocation in an interference limited Rayleigh and Nakagami fading wireless channels is the minimum power vector in the total strategy spaces of active users in the system. Power was more effectively and more simply allocated according to this proposed non-centralized algorithm than the centralized algorithm in [5].

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