INTERNATIONAL JOURNAL OF COMPUTERS COMMUNICATIONS & CONTROL
Online ISSN 1841-9844, ISSN-L 1841-9836, Volume: 16, Issue: 3, Month: June, Year: 2021
Article Number: 4214, https://doi.org/10.15837/ijccc.2021.3.4214

CCC Publications

Multi-attribute Group Decision Making Method with Unknown Attribute Weights Based on the Q-rung Orthopair Uncertain Linguistic Power Muirhead Mean Operators

H.M. Zhao, R.T. Zhang, A. Zhang, X.M. Zhu

Hongmei Zhao, Runtong Zhang
School of Economics and Management
Beijing Jiaotong University, Beijing, China
No.3 Shangyuancun Haidian District Beijing 100044 P. R. China
E-mail: zhaohongmei81@163.com, rtzhang@bjtu.edu.cn

Ao Zhang, Xiaomin Zhu*
School of Mechanical, Electronic, and Control Engineering
Beijing Jiaotong University, Beijing, China
No.3 Shangyuancun Haidian District Beijing 100044 P. R. China
E-mail: aozhang@bjtu.edu.cn, xmzhu@bjtu.edu.cn
*Corresponding author: xmzhu@bjtu.edu.cn

Abstract

Q-rung orthopair uncertain linguistic sets (q-ROULSs) are a powerful tool for describing ambiguity and uncertainty of linguistic information. In this study, considering that in most multi-attribute group decision making (MAGDM) problems, not only the quantitative evaluation information of decision makers but also the qualitative evaluation opinions should be considered. Therefore, we develop a novel MAGDM method with unknown attribute weights under the q-rung orthopair uncertain linguistic environments. We firstly propose the cross-entropy of q-ROULSs, which is utilized to solve the optimal attribute weights by a linear programming model. In order to effectively summarize the unclear language information of q-ROULSs, we extend the power Muirhead mean (PMM) operator to q-ROULSs, and propose a family of q-rung othpair uncertain linguistic power Muirhead mean (q-ROULPMM) operators. The advantage of the PMM operator is that it not only mitigates the adverse effects of too high or too low attribute values on the results, but also takes into account the interrelationships between attribute values. At the same time, some ideal properties and special cases of the q-ROULPMM operator are also studied. Further, a new method based on the proposed cross-entropy and aggregation operators is developed for solving the MAGDM problem under q-ROULSs. Finally, we carried out numerical experiments to prove the effectiveness and superiority of the method.

Keywords: q-rung orthopair uncertain linguistic set, cross -entropy, attribute weights, q-rung orthopair uncertain linguistic power Muirhead mean, multi-attribute group decision making.

1 Introduction

The multi-attribute group decision problem (MAGDM) is one of the most important branches of modern decision theory and has received increasing attention in the past few years. MAGDM can accomplish the selection of the best one among many alternatives according to a series of attribute indicators by multiple decision makers. In practical MAGDM problem, one of the most important difficulties is the representation of attribute values in uncertain decision environments. In 1965, Zadeh [1] initialized the concept of fuzzy sets (FSs), which open a new unchartered territory for dealing with the vague and uncertain information by a membership degree function. Later, two significant extensions of fuzzy sets, i.e. intuitionistic fuzzy sets (IFSs) and Pythagorean fuzzy sets (PFSs), have also been proposed in [2] and [3], which are characterized by both membership and non-membership to describe the uncertainty and hesitancy of information more accurately. Since the introduction of IFSs and PFSs, they have attracted widespread attention among scholars and are widely used in medical diagnosis [4],[5], pattern recognition [6, 7, 8], data mining [9, 10] and MAGDM [11, 12, 13, 14].

Although the IFSs and PFSs are powerful, they require membership and non-membership degrees to satisfy some certain constraints. Specifically, IFSs ask that the sum of membership and non-membership degrees is less than one, and PFSs specify the square sum of membership and non-membership degrees is less than or equal to one. This feature limits the ability of IFSs and PFSs in describing the fuzzy and uncertain information. For example, decision makers maybe define the degree of membership and non-membership as (0.8, 0.7), while it is not valid to IFSs and PFSs. In order to solve these problems effectively, Yager [15] proposed the concept of q-rung orthopair fuzzy sets (q-ROFSs), in which a parameter q greater than or equal to one is defined to adjust the expressed range of fuzzy information flexibly. It is obvious that q-ROFSs are more applicable than IFSs and PFSs when copying with fuzzy and uncertainties. In recent, many studies on q-ROFSs have been undertaken from both theoretical and practical aspects. For example, some operational laws of q-rung orthopair fuzzy numbers, such as Algebraic, Einstein, Hamacher, have been defined in [16, 17, 18]. Du [19] proposed some Minkowski-type distance measures for the decision-making application of q-ROFSs. Liang et al. [20] proposed the q-rung orthopair fuzzy cross-entropy to identify the fuzzy measures between q-rung orthopair fuzzy numbers(q-ROFNs). On the other hand, some traditional decision making methods, e.g. TODIM[21], TOPSIS[22],and MABAC[23], have also been applied in the q-rung oethoapir fuzzy environment. In addition, a large number of q-rung orthopair fuzzy aggregation operators have also developed for solving the MAGDM problem[24, 25].

In addition to the quantitative assessments, we must consider the semantic evaluation opinions given by decision makers. The linguistic variables (LVs) are considered as an ideal solution provider to cope with the semantic assessments, such as 'good', 'fair', 'worse', etc. [26]. However, some sematic opinions cannot be described by a single LV, For instance, decision makers maybe provide an assessment which is lower than "good" but higher than 'fair'. Therefore, Xu [27] put forward to the uncertain linguistic variables (ULVs) that leverage two linguistic terms to represent a semantic interval. However, an pivotal shortcoming of LVs and ULVs is that they cannot describe the decision maker's reliability and uncertainty for a given linguistic evaluation. To remedy this bottleneck, Liu and Qin [28] utilized the membership and non-membership degrees of the IFS to represent the hesitancy and uncertainty of the ULV, and proposed the intuitionistic uncertain linguistic variables (IULVs). Further, some extensions of IULVs, that combine ULVs with some more advanced fuzzy sets, have also been proposed in [29],[30],[31]. Among them, q-rung orthopair uncertain linguistic sets (q-ROULSs) [31], [32], that combine the q-ROFSs with ULVs, not only enable an intuitionistic evaluation for hesitancy and uncertainty for ULVs, but also accomplish the flexible adjustment of the indication range of decision information.

Another challenging problem in MAGDM is the aggregation of attributes information and the ranking of alternatives. At present, a variety of aggregation operators have been studied and achieved the significant success in MAGDM problems[33, 34, 35, 36]. In view of the increased complexity of actual decision-making problems, we may consider the following three issues when choosing the best alternative. (1) The evaluation values of attributes provided by decision makers is too high or too low, which have a negative impact on the final result. The PA operator proposed by Yager can better avoid this problem as it allows to discount outliers according to automatically assigning a power weight to each attribute. (2) The attributes of alternatives are usually related, so we need to consider the various relationship between the attributes. Hence, a variety of aggregation operators are proposed to solve this problem, such as Bonferroni mean (BM) [34], Heronian mean (HM) [35], Maclaurin Symmetric Mean (MSM) [37] and Muirhead mean (MM) [36], and so on. It is worthy stressing that MM has obvious advantages over other several operators, as it can consider the interrelationship between all aggregated values, meanwhile, it can reduce into BM and MSM by adjusting its parameters vector. (3) In the case of various relationships between attributes, there are outlier assessments at the same time. To solve the above two situations simultaneously, the power Muirhead mean(PMM) operator is proposed in [38]. They inherit the advantages of PA and MM operators at the same time, and is widely used to solve the various MAGDM problem [39, 40].

Although a variety of aggregation operators have been proposed to solve the MAGDM problem, the studies on MAGDM based on the q-rung orthopair uncertain linguistic aggregation operators are still scarce. For example, Liu et al. [41] proposed the q-rung orthopair uncertain linguistic weighted average(WA) operator and q-rung orthopair uncertain linguistic ordered weighted average (OWA) operator for the decision-making application. Liu et al.[42] defined the q-rung orthopair uncertain linguistic partitioned Bonferroni mean (PBM) operator to solve this situation where some attributes are related, while other attributes are not related. However, the aforementioned methods fail to reflect the interrelationship between all arguments, and cannot automatically eliminate the outlier assessments on aggregation results at the same time. In addition, the attribute weights are directly given by decision makers in existing q-rung orthopair uncertain linguistic MAGDM methods. It is obvious that this strategy cannot guarantee the rationality of weight information. Therefore, this paper develops a MAGDM method based on q-rung orthopair uncertain PMM operator with unknown attributes weights. In order to do this, we firstly define the cross-entropy of q-ROULSs, which is utilized to obtain the optimal weight vector of attributes based on a linear programming model. Secondly, we first propose q-rung orthopair uncertain linguistic PMM (q-ROULPMM) operator and its weighted form to summarize the decision maker's preference information and determine the best choice. Then, a new MAGDM method are also developed based on the proposed cross-entropy and aggregation operators in q-ROULSs. Finally, a numerical example is provided to demonstrate the effectiveness and superiority of the proposed method.

The rest of this article is organized as follows. Section 2 reviews the basic concepts and proposes the cross entropy of q-ROULSs. Section 3 elaborates the q-ROULPMM operator and its weighted form. Section 4 introduces a new MAGDM method. Section 5 describes the performance and superiority of the proposed method by a numerical instance as well as comparative analysis. The conclusion is given in Section 6.

2 Preliminaries

2.1 Q-rung orthopair uncertain linguistic sets

Definition 1 [15] Let X be a ordinary fixed set, a q-rung orthopair fuzzy set (q-ROFS) A on X is defined:

$$A = \{x, \mu_A(x), v_A(x) | x \in X\} \, (q \geq 1) \tag{1}$$

where $\mu_A(x)$ and $v_A(x)$ respectively represent the membership and non-membership degrees satisfying $\mu_A(x) \in [0,1]$, $v_A(x) \in [0,1]$ and $0 \leq \mu_A^q(x) + v_A^q(x) \leq 1$. For convenience, the pair $(\mu_A(x), v_A(x))$ is called as a q-rung orthopair fuzzy number (q-ROFN), which can be denoted by $A = (\mu_A, v_A)$.

Let $S = \{s_i | i = 0, 1, \ldots, g\}$ be a linguistic term set (LTS) with odd cardinality, where s_i represents the i-th linguistic variable (LV) of S, $g+1$ is the cardinality of S, which usually is set to a small odd number, such as 5, 7, 9. For the linguistic set S, the following conditions should be satisfied:

1) Orderliness: $s_i > s_j$, if $i > j$;

2) Negative operator: $\text{Neg}(s_i) = s_j$, where $j = g - i$;

3) Maximize and minimize operator: $\max(s_i, s_j) = s_i$, if $s_i \geq s_j$, $\min(s_i, s_j) = s_i$, if $s_i \leq s_j$.

Furthermore, Xu [27] developed the concept of uncertain linguistic variable (ULV) to preserve all the given information.

Definition 2 Let $\tilde{s} = [s_\theta, s_\tau]$, $s_\theta, s_\tau \in \tilde{S}$ and $0 < \theta < \tau$, $\tilde{S} = \{s_\alpha \,| s_0 \le s_\alpha \le s_t, \alpha \in [0,t]\}$ be a continuous term set, s_θ, s_τ represent the lower limit and upper limit of $\tilde{s}$, respectively, then the $\tilde{s}$ is an ULV.

Let $\tilde{s}_1 = [s_{\theta_1}, s_{\tau_1}]$, $s_2 = [\tilde{s}_{\theta_2}, s_{\tau_2}]$ be two any ULVs, λ is an positive real number, the operational laws are showed as follows:

1. $\tilde{s}_1 \oplus \tilde{s}_2 = [s_{\theta_1}, s_{\tau_1}] \oplus [s_{\theta_1}, s_{\tau_2}] = [s_{\theta_1+\theta_2}, s_{\tau_1+\tau_2}]$
2. $\tilde{s}_1 \otimes \tilde{s}_2 = [s_{\theta_1}, s_{\tau_1}] \otimes [s_{\theta_2}, s_{\tau_2}] = [s_{\theta_1\times\theta_2}, s_{\tau_1\times\tau_2}]$
3. $\lambda\tilde{s}_1 = \lambda[s_{\theta_1}, s_{\tau_1}] = [s_{\lambda\theta_1}, s_{\lambda\tau_1}]$
4. $\tilde{s}_1^\lambda = ([s_{\theta_1}, s_{\tau_1}])^\lambda = \left[s_{(\theta_1)^\lambda}, s_{(\tau_1)^\lambda}\right]$

Although the ULVs express the semantic information conveniently, they are incapable of expressing the hesitancy and uncertainty of semantic information intuitively. To remedy the above deficiency, q-ROULSs [31], [32] that combine the ULVs and q-ROFSs, leverage the membership and non-membership degree to describe the hesitant degree of the ULVs.

Definition 3 Let X be an ordinary fixed set, then a q-rung orthopair uncertain linguistic set A defined on X is expressed as

$$A = \left\{\left\langle x\left[\left[s_{\theta(x)}, s_{\tau(x)}\right], (u_A(x), v_A(x))\right]\right\rangle |x \in X\right\} (q \ge 1) \tag{2}$$

where $s_{\theta(x)}, s_{\tau(x)} \in \tilde{S}$ is the ULV of x, $\tilde{S}$ be a continuous linguistic term set, $\mu_A(x)$ and $v_A(x)$ represent the membership and non-membership degrees of x to ULV $\left[s_{\theta(x)}, s_{\tau(x)}\right]$, where $\mu_A(x), v_A(x) \in [0,1]$ and $0 \le \mu_A^q(x) + v_A^q(x) \le 1$. For convenience, we call $\left\langle\left[s_{\theta(x)}, s_{\tau(x)}\right], (\mu_A(x), v_A(x))\right\rangle$ as a q-rung orthopair uncertain linguistic value (q-ROULV), which can be denoted by $\alpha = \langle[s_\theta, s_\tau], (\mu_A, v_A)\rangle$.

Definition 4 Let $\alpha_1 = \langle[s_{\theta_1}, s_{\tau_1}], (u_1, v_1)\rangle$, $\alpha_2 = \langle[s_{\theta_2}, s_{\tau_2}], (u_2, v_2)\rangle$ be any two q-ROULVs, and λ be a positive real number, then

1. $\alpha_1 \oplus \alpha_2 = \left\langle[s_{\theta_1+\theta_2}, s_{\tau_1+\tau_2}], \left((\mu_1^q + \mu_2^q - \mu_1^q\mu_2^q)^{1/q}, v_1v_2\right)\right\rangle$,
2. $\alpha_1 \otimes \alpha_2 = \left\langle[s_{\theta_1*\theta_2}, s_{\tau_1*\tau_2}], \left(\mu_1\mu_2, (v_1^q + v_2^q - v_1^qv_2^q)^{1/q}\right)\right\rangle$,
3. $\lambda\alpha_1 = \left\langle[s_{\lambda*\theta_1}, s_{\lambda*\tau_1}], \left(\left(1-(1-\mu_1^q)^\lambda\right)^{1/q}, v_1^\lambda\right)\right\rangle$,
4. $(\alpha_1)^\lambda = \left\langle\left[s_{\theta_1^\lambda}, s_{\tau_1^\lambda}\right], \left(\mu_1^\lambda, \left(1-(1-v_1^q)^\lambda\right)^{1/q}\right)\right\rangle$.

Definition 5 Let $\alpha = \langle[s_\theta, s_\tau], (\mu_A, v_A)\rangle$ be a q-ROULV, then the expected value $E(\alpha)$ of α is defined as $E(\alpha) = \frac{\theta+\tau}{4}(\mu^q + 1 - v^q)$, and the accuracy function $H(\alpha)$ of α is defined as $H(\alpha) = \frac{\theta+\tau}{2}(\mu^q + v^q)$. For any two q-ROULVs α_1 and α_2, we have

1. If $E(\alpha_1) > E(\alpha_2)$, then$\alpha_1 > \alpha_2$,
2. If $E(\alpha_1) = E(\alpha_2)$, then
   (a) if $H(\alpha_1) > H(\alpha_2)$, then $\alpha_1 > \alpha_2$.
   (b) if $H(\alpha_1) = H(\alpha_2)$, then $\alpha_1 = \alpha_2$.

2.2 Cross entropy of q-ROULVs

The cross-entropy measure is an important operation to measure the relation between two sets or objects. It is used to calculate the divergence between two probability distributions or two random variables. Recently, Liang et al. [20] defined the q-rung orthopair fuzzy cross-entropy to identify the fuzzy measures between q-ROFNs.

Definition 6 [20] Let $a_1 = (\mu_1, v_1)$ and $a_2 = (\mu_2, v_2)$ be two q-ROFNs, then the cross-entropy $CE(a_1, a_2)$ of a_1 and a_2 can be defined as follows

$$\begin{aligned} CE(a_1,a_2) = \frac{1}{1-2^{1-p}} \Bigg(& \frac{(\mu_1)^{pq}+(\mu_2)^{pq}}{2} - \left(\frac{(\mu_1)^q+(\mu_2)^q}{2}\right)^p + \frac{(v_1)^{pq}+(v_2)^{pq}}{2} \\ & - \left(\frac{(v_1)^q+(v_2)^q}{2}\right)^p + \frac{(\pi_1)^{pq}+(\pi_2)^{pq}}{2} - \left(\frac{(\pi_1)^q+(\pi_2)^q}{2}\right)^p \Bigg) \end{aligned} \tag{3}$$

where π_1 and π_2 are the indeterminacy degree of a_1 and a_2, respectively.

Although the cross-entropy has achieved breakthrough successes in various fuzzy environments, the studies on cross-entropy are still a blank under q-rung orthopair uncertain linguistic environments. Therefore, the cross-entropy of q-ROULVs is presented herein.

Definition 7 Let $\alpha_1 = \langle [s_{\theta_1}, s_{\tau_1}], (\mu_1, v_1)\rangle$ and $\alpha_2 = \langle [s_{\theta_2}, s_{\tau_2}], (\mu_2, v_2)\rangle$, $g+1$ is the cardinality of linguistic term set, then the cross-entropy of α_1 and α_2 is defined as:

$$\begin{aligned} CE(\alpha_1,\alpha_2) = & \frac{\theta_1+\tau_1}{2g}\ln\frac{2(\theta_1+\tau_1)}{\theta_1+\tau_1+\theta_2+\tau_2} + \frac{\tau_1-\theta_1}{g}\ln\frac{2(\theta_1-\tau_1)}{(\theta_1-\tau_1)+(\theta_2-\tau_2)} \\ & + (1-\frac{\theta_1+\tau_1}{2g})\ln\frac{4g-2(\theta_1+\tau_1)}{4g-(\theta_1+\tau_1+\theta_2+\tau_2)} \\ & + (1+\frac{\theta_1-\tau_1}{g})\ln\frac{2(g+(\theta_1-\tau_1))}{2g+(\theta_1-\tau_1)+(\theta_2-\tau_2)} \\ & + \mu_1^q\ln\frac{2\mu_1^q}{\mu_1^q+\mu_2^q} + v_1^q\ln\frac{2v_1^q}{v_1^q+v_2^q} \end{aligned} \tag{4}$$

Theorem 1 Let $\alpha_1 = \langle [s_{\theta_1}, s_{\tau_1}], (\mu_1, v_1)\rangle$ and $\alpha_2 = \langle [s_{\theta_2}, s_{\tau_2}], (\mu_2, v_2)\rangle$, be any two q-ROULVs, then cross-entropy $CE(\alpha_1, \alpha_2)$ satisfies the following properties:

1) $CE(\alpha_1, \alpha_2) \geq 0$,

2) $CE(\alpha_1, \alpha_2) = 0$, if $\alpha_1 = \alpha_2$,

3) $CE(\alpha_1, \alpha_2) = CE(\alpha_1^c, \alpha_2^c)$, where $\alpha_i^c = \langle [g - s_{\tau_i}, g - s_{\theta_i}], (v_i, \mu_i)\rangle$

2.3 Power average operator and Muirhead mean operator

The PA operator was proposed by Yager for crisp numbers. The prominent characteristic of PA is that it allows the weighting vector to depend on the input arguments and evaluation.

Definition 8 Let $\tilde{a}_i (i = 1, 2, \cdots, n)$ is a collection of nonnegative real numbers, then the power average (PA) operator is defined as:

$$PA(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \sum_{i=1}^{n} \frac{(1+T(\tilde{a}_i))\,\tilde{a}_i}{\sum\limits_{j=1}^{n}(1+T(\tilde{a}_j))}, \tag{5}$$

where $T(\tilde{a}_i) = \sum_{j=1, j\neq i}^{n} Sup(\tilde{a}_i, \tilde{a}_j)$ and $Sup(\tilde{a}_i, \tilde{a}_j)$ indicates the support for a_i from a_j, which satisfies the following properties:

1. $Sup\left(\tilde{a}_i, \tilde{a}_j\right) \in [0,1]$,
2. $Sup\left(\tilde{a}_i, \tilde{a}_j\right) = Sup\left(\tilde{a}_j, \tilde{a}_i\right)$,
3. $Sup\left(\tilde{a}_i, \tilde{a}_j\right) \geq Sup\left(\tilde{a}_s, \tilde{a}_t\right)$, if $d(\tilde{a}_i, \tilde{a}_j) < d(\tilde{a}_s, \tilde{a}_t)$, where $d(\tilde{a}_i, \tilde{a}_j)$ is the distance between $\tilde{a}_i$ and $\tilde{a}_j$.

The MM was an aggregation technology proposed by Muirhead for crisp numbers, they can deals with the interrelationship among all arguments.

Definition 9 Let $a_i\,(i = 1, 2, ..., n)$be a collection of crisp numbers and $K{=}\,(k_1, k_2, ..., k_n) \in R^n$, then the Muirhead mean (MM) operator is defined as

$$MM^K\left(a_1, a_2, ..., a_n\right) = \left(\frac{1}{n!}\sum_{\vartheta\in S_n}\prod_{j=1}^{n} a_{\vartheta(j)}^{P_j}\right)^{\frac{1}{\sum_{j=1}^{n} P_j}}, \tag{6}$$

where $\vartheta\,(j)\,(j = 1, 2, ..., n)$ is any permutation of $(1, 2, ..., n)$,and S_n is the collection of all permutation of $(1, 2, ..., n)$.

3 Some new q-rung orthopair uncertain linguistic aggregation operators

3.1 The q-rung orthopair uncertain linguistic power Muirhead mean operator

Definition 10 Let $\alpha_j = \left\langle\left[s_{\theta_j}, s_{\tau_j}\right], (\mu_j, v_j)\right\rangle (j = 1, 2, ..., n)$ be a collection q-ROULVs and $K = (k_1, k_2, ..., k_n) \in R^n$ be a set of parameters. If

$$q - ROULPMM^K\left(\alpha_1, \alpha_2, ..., \alpha_n\right) = \left(\frac{1}{n!}\sum_{\vartheta\in S_n}\prod_{j=1}^{n}\left(\frac{n\left(1+T\left(\alpha_{\vartheta(j)}\right)\right)}{\sum_{i=1}^{n}\left(1+T\left(\alpha_i\right)\right)}\alpha_{\vartheta(j)}\right)^{k_j}\right)^{\frac{1}{\sum_{i=1}^{n} k_j}}, \tag{7}$$

where $T\left(\alpha_j\right) = \sum_{i=1, j\neq i}^{n} Sup\left(\alpha_i, \alpha_j\right)$, $Sup\left(\alpha_i, \alpha_j\right)$ represents the support degree for α_i and α_j, which satisfies the following properties:

1. $Sup\left(\alpha_i, \alpha_j\right) \in [0, 1]$
2. $Sup\left(\alpha_i, \alpha_j\right) = Sup\left(\alpha_j, \alpha_i\right)$
3. $Sup\left(\alpha_i, \alpha_j\right) > Sup\left(\alpha_s, \alpha_t\right)$, if $d\left(\alpha_i, \alpha_j\right) < d\left(\alpha_s, \alpha_t\right)$, where $d\left(\alpha_i, \alpha_j\right)$ denotes the distance between α_i and α_j.

Further, let

$$\omega_j = \frac{\left(1+T\left(\alpha_j\right)\right)}{\sum_{i=1}^{n}\left(1+T\left(\alpha_i\right)\right)}, \tag{8}$$

then we can obtain the simplified form of Eq.(7):

$$q - ROULPMM^K\left(\alpha_1, \alpha_2, \ldots, \alpha_n\right) = \left(\frac{1}{n!}\sum_{\vartheta\in S_n}\prod_{j=1}^{n}\left(n\omega_{\vartheta(j)}\alpha_{\vartheta(j)}\right)^{k_j}\right)^{\frac{1}{\sum_{j=1}^{n} k_j}} \tag{9}$$

where $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ is the power weighting vector (PWV) satisfying $\omega_i \in [0, 1]$ and $\sum_{i=1}^{n}\omega_i = 1$

According to the operations of Definition (4), we can get q-ROULPMM satisfies following theorems.

Theorem 2 Let $\alpha_j = \left\langle [s_{\theta_j}, s_{\tau_j}], (\mu_j, v_j) \right\rangle (j = 1, 2, ..., n)$ be a collection q-ROULVs and $K = (k_1, k_2, ..., k_n) \in R^n$ be a set of parameters, then the aggregated value by q-ROULPMM is still a q-ROULV, and

$$q - ROULPMM^K (\alpha_1, \alpha_2, ..., \alpha_n) = \left\langle \left[s_{\left(\frac{1}{n!} \sum\limits_{\vartheta \in S_n} \prod\limits_{j=1}^{n} (n\omega_j \theta_{\vartheta(j)})^{k_j} \right)^{\frac{1}{\sum_{j=1}^{n} k_j}}}, s_{\left(\frac{1}{n!} \sum\limits_{\vartheta \in S_n} \prod\limits_{j=1}^{n} (n\omega_j \tau_{\vartheta(j)})^{k_j} \right)^{\frac{1}{\sum_{j=1}^{n} k_j}}} \right], \right.$$
$$\left(\left(\left(1 - \left(\prod_{\vartheta \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \left(1 - \mu_{\vartheta(j)}^q \right)^{n\omega_j} \right)^{k_j} \right) \right)^{\frac{1}{n!}} \right)^{1/q} \right)^{\frac{1}{\sum_{j=1}^{n} k_j}}, \right. \tag{10}$$
$$\left. \left(1 - \left(1 - \prod_{\vartheta \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - v_{\vartheta(j)}^{qn\omega_j} \right)^{r_j} \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} r_j}} \right)^{1/q} \right) \right\rangle.$$

Proof. According to Definition (4), we have

$$\mathrm{n}\omega_j \alpha_{\vartheta(j)} = \left\langle \left[s_{n\omega_j \theta_{\vartheta(j)}}, s_{n\omega_j \tau_{\vartheta(j)}} \right], \left(\left(1 - \left(1 - \mu_{\vartheta(j)}^q \right)^{n\omega_j} \right)^{1/q}, v_{\vartheta(j)}^{n\omega_j} \right) \right\rangle.$$

and

$$\left(\mathrm{n}\omega_j \alpha_{\vartheta(j)} \right)^{k_j} = \left\langle \left[s_{(n\omega_j \theta_{\vartheta(j)})^{k_j}}, s_{(n\omega_j \tau_{\vartheta(j)})^{k_j}} \right], \right.$$
$$\left. \left(\left(\left(1 - \left(1 - \mu_{\vartheta(j)}^q \right)^{n\omega_j} \right)^{1/q} \right)^{k_j}, \left(1 - \left(1 - v_{\vartheta(j)}^{qn\omega_j} \right)^{k_j} \right)^{1/q} \right) \right\rangle.$$

Thus, we can obtain

$$\prod_{j=1}^{n} \left(\mathrm{n}\omega_j \alpha_{\vartheta(j)} \right)^{k_j} = \left\langle \left[s_{\prod\limits_{j=1}^{n} (n\omega_j \theta_{\vartheta(j)})^{k_j}}, s_{\prod\limits_{j=1}^{n} (n\omega_j \tau_{\vartheta(j)})^{k_j}} \right], \right.$$
$$\left. \left(\prod_{j=1}^{n} \left(\left(1 - \left(1 - \mu_{\vartheta(j)}^q \right)^{n\omega_j} \right)^{1/q} \right)^{k_j}, \left(1 - \prod_{j=1}^{n} \left(1 - v_{\vartheta(j)}^{qn\omega_j} \right)^{k_j} \right)^{1/q} \right) \right\rangle.$$

and

$$\sum_{\vartheta \in S_n} \prod_{j=1}^{n} \left(\mathrm{n}\omega_j \alpha_{\vartheta(j)} \right)^{k_j} = \left\langle \left[s_{\sum\limits_{\vartheta \in S_n} \prod\limits_{j=1}^{n} (n\omega_j \theta_{\vartheta(j)})^{k_j}}, s_{\sum\limits_{\vartheta \in S_n} \prod\limits_{j=1}^{n} (n\omega_j \tau_{\vartheta(j)})^{k_j}} \right], \right.$$
$$\left(\left(1 - \prod_{\vartheta \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \left(1 - \mu_{\vartheta(j)}^q \right)^{n\omega_j} \right)^{k_j} \right) \right)^{1/q}, \right.$$
$$\left. \left. \prod_{\vartheta \in S_n} \left(\left(1 - \prod_{j=1}^{n} \left(1 - v_{\vartheta(j)}^{qn\omega_j} \right)^{k_j} \right)^{1/q} \right) \right) \right\rangle.$$

Thus,

$$\frac{1}{n!}\sum_{\vartheta\in S_n}\prod_{j=1}^{n}\left(\mathrm{n}\omega_j\alpha_{\vartheta(j)}\right)^{k_j}=\left\langle\left[s_{\frac{1}{n!}\sum_{\vartheta\in S_n}\prod_{j=1}^{n}\left(n\omega_j\theta_{\vartheta(j)}\right)^{k_j}},s_{\frac{1}{n!}\sum_{\vartheta\in S_n}\prod_{j=1}^{n}\left(n\omega_j\tau_{\vartheta(j)}\right)^{k_j}}\right],\right.$$
$$\left(\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\mu_{\vartheta(j)}^{q}\right)^{n\omega_j}\right)^{k_j}\right)\right)^{\frac{1}{n!}}\right)^{1/q},\right.$$
$$\left.\left.\left(\prod_{\vartheta\in S_n}\left(\left(1-\prod_{j=1}^{n}\left(1-v_{\vartheta(j)}^{qn\omega_j}\right)^{k_j}\right)^{1/q}\right)\right)^{\frac{1}{n!}}\right)\right\rangle.$$

Therefore,

$$\left(\frac{1}{n!}\sum_{\vartheta\in S_n}\prod_{j=1}^{n}\left(\mathrm{n}\omega_j\alpha_{\vartheta(j)}\right)^{k_j}\right)^{\frac{1}{\sum_{j=1}^{n}k_j}}=$$
$$\left\langle\left[s_{\left(\frac{1}{n!}\sum_{\vartheta\in S_n}\prod_{j=1}^{n}\left(n\omega_j\theta_{\vartheta(j)}\right)^{k_j}\right)^{\frac{1}{\sum_{j=1}^{n}k_j}}},s_{\left(\frac{1}{n!}\sum_{\vartheta\in S_n}\prod_{j=1}^{n}\left(n\omega_j\tau_{\vartheta(j)}\right)^{k_j}\right)^{\frac{1}{\sum_{j=1}^{n}k_j}}}\right],\right.$$
$$\left(\left(\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\mu_{\vartheta(j)}^{q}\right)^{n\omega_j}\right)^{k_j}\right)\right)^{\frac{1}{n!}}\right)^{1/q}\right)^{\frac{1}{\sum_{j=1}^{n}k_j}},\right.$$
$$\left.\left.\left(1-\left(1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-v_{\vartheta(j)}^{qn\omega_j}\right)^{k_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}k_j}}\right)^{1/q}\right)\right\rangle.$$

Theorem 3 (Idempotency) Let $\alpha_j=\left\langle\left[s_{\theta_j},s_{\tau_j}\right],(\mu_j,v_j)\right\rangle$ $(j=1,2,...,n)$ be a collection q-ROULVs, and $\alpha_j=\alpha=\langle[s_\theta,s_\tau],(\mu,v)\rangle$ for j $=1,2,...,n$. Then,

$$q-ROULPMM^K\left(\alpha_1,\alpha_2,...,\alpha_n\right)=\alpha \tag{11}$$

Proof. As $\alpha_j=\alpha=\langle[s_\theta,s_\tau],(\mu,v)\rangle$ holds for all j, we have $Sup\left(\alpha_j,\alpha_i\right)=1$ for $i,j=1,2,\ldots,n$. we can obtain $\omega_j=1/n$ for all j. Further, we can get

$$q-ROULPMM^K\left(\alpha,\alpha,...,\alpha\right)=\left\langle\left[s_{\left(\frac{1}{n!}\sum_{\vartheta\in S_n}\prod_{j=1}^{n}\left(n\frac{1}{n}\theta\right)^{k_j}\right)^{\frac{1}{\sum_{j=1}^{n}k_j}}},s_{\left(\frac{1}{n!}\sum_{\vartheta\in S_n}\prod_{j=1}^{n}\left(n\frac{1}{n}\tau\right)^{k_j}\right)^{\frac{1}{\sum_{j=1}^{n}k_j}}}\right],\right.$$
$$\left(\left(\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-(1-\mu^q)^{n\frac{1}{n}}\right)^{k_j}\right)\right)^{\frac{1}{n!}}\right)^{1/q}\right)^{\frac{1}{\sum_{j=1}^{n}k_j}},\right.$$
$$\left.\left.\left(1-\left(1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-v^{qn\frac{1}{n}}\right)^{k_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}k_j}}\right)^{1/q}\right)\right\rangle$$
$$=\left\langle\left[s_{\frac{1}{n!}n!\theta},s_{\frac{1}{n!}n!\tau}\right],\left((\mu^q)^{1/q},(v^q)^{1/q}\right)\right\rangle=\alpha. \tag{12}$$

which completes the proof of Theorem (3).

Theorem 4 (Boundedness) Let $\alpha_j = \left\langle \left[s_{\theta_j}, s_{\tau_j} \right], (\mu_j, v_j) \right\rangle (j = 1, 2, ..., n)$, $\alpha^- = \min (\alpha_1, \alpha_2, ..., \alpha_n)$ and $\alpha^+ = \max (\alpha_1, \alpha_2, ..., \alpha_n)$, then

$$\alpha^- \leq q - ROULPMM^K (\alpha_1, \alpha_2, ..., \alpha_n) \leq \alpha^+ \tag{13}$$

where $\alpha^- = \langle [s_{\theta^-}, s_{\tau^-}], (a, b) \rangle$, and $\alpha^+ = \langle [s_{\theta^+}, s_{\tau^+}], (c, d) \rangle$. and

$$s_{\theta^-} = s_{\left(\frac{1}{n!} \sum\limits_{\vartheta \in S_n} \prod\limits_{j=1}^{n} \left(n \frac{1}{n} \theta^- \right)^{k_j} \right)^{\frac{1}{\sum_{j=1}^{n} k_j}}}, \; s_{\tau^-} = s_{\left(\frac{1}{n!} \sum\limits_{\vartheta \in S_n} \prod\limits_{j=1}^{n} \left(n \frac{1}{n} \tau^- \right)^{k_j} \right)^{\frac{1}{\sum_{j=1}^{n} k_j}}},$$

$$a = \left(\left(1 - \left(\prod_{\vartheta \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - (1 - a^q)^{n\omega_j} \right)^{k_j} \right) \right)^{\frac{1}{n!}} \right)^{1/q} \right)^{\frac{1}{\sum_{j=1}^{n} k_j}},$$

and

$$b = \left(1 - \left(1 - \prod_{\vartheta \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - b^{qn\omega_j} \right)^{k_j} \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} k_j}} \right)^{1/q}.$$

Meanwhile,

$$s_{\theta^+} = s_{\left(\frac{1}{n!} \sum\limits_{\vartheta \in S_n} \prod\limits_{j=1}^{n} \left(n \frac{1}{n} \theta^+ \right)^{k_j} \right)^{\frac{1}{\sum_{j=1}^{n} k_j}}}, \; s_{\tau^+} = s_{\left(\frac{1}{n!} \sum\limits_{\vartheta \in S_n} \prod\limits_{j=1}^{n} \left(n \frac{1}{n} \tau^+ \right)^{k_j} \right)^{\frac{1}{\sum_{j=1}^{n} k_j}}},$$

$$c = \left(\left(1 - \left(\prod_{\vartheta \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - (1 - c^q)^{n\omega_j} \right)^{k_j} \right) \right)^{\frac{1}{n!}} \right)^{1/q} \right)^{\frac{1}{\sum_{j=1}^{n} k_j}},$$

and

$$d = \left(1 - \left(1 - \prod_{\vartheta \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - d^{qn\omega_j} \right)^{k_j} \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} k_j}} \right)^{1/q}.$$

Proof. First, for α^- we can have

$$\begin{aligned} n\omega_j \alpha_{\vartheta(j)} &= \left\langle \left[s_{n\omega_j \theta_{\vartheta(j)}}, s_{n\omega_j \tau_{\vartheta(j)}} \right], \left(\left(1 - \left(1 - \mu_{\vartheta(j)}^q \right)^{n\omega_j} \right)^{1/q}, v_{\vartheta(j)}^{n\omega_j} \right) \right\rangle \\ &\geq \left\langle \left[s_{n\omega_j \theta^-}, s_{n\omega_j \tau^-} \right], \left(\left(1 - (1 - a^q)^{n\omega_j} \right)^{1/q}, b^{n\omega_j} \right) \right\rangle. \end{aligned}$$

and

$$\begin{aligned} \left(n\omega_j \alpha_{\vartheta(j)} \right)^{k_j} &= \left\langle \left[s_{\left(n\omega_j \theta_{\vartheta(j)} \right)^{k_j}}, s_{\left(n\omega_j \tau_{\vartheta(j)} \right)^{k_j}} \right], \right. \\ &\left. \left(\left(\left(1 - \left(1 - \mu_{\vartheta(j)}^q \right)^{n\omega_j} \right)^{1/q} \right)^{k_j}, \left(1 - \left(1 - v_{\vartheta(j)}^{qn\omega_j} \right)^{k_j} \right)^{1/q} \right) \right\rangle \\ &\geq \left\langle \left[s_{(n\omega_j \theta^-)^{k_j}}, s_{(n\omega_j \tau^-)^{k_j}} \right], \left(\left(\left(1 - (1 - a^q)^{n\omega_j} \right)^{1/q} \right)^{k_j}, \left(1 - \left(1 - b^{qn\omega_j} \right)^{k_j} \right)^{1/q} \right) \right\rangle. \end{aligned}$$

Further,

$$\prod_{j=1}^{n}\left(\mathrm{n}\omega_j\alpha_{\vartheta(j)}\right)^{k_j}=\left\langle\left[s_{\prod\limits_{j=1}^{n}\left(n\omega_j\theta_{\vartheta(j)}\right)^{k_j}},s_{\prod\limits_{j=1}^{n}\left(n\omega_j\tau_{\vartheta(j)}\right)^{k_j}}\right],\left(\prod_{j=1}^{n}\left(\left(1-\left(1-\mu_{\vartheta(j)}^{q}\right)^{n\omega_j}\right)^{1/q}\right)^{k_j},\left(1-\prod_{j=1}^{n}\left(1-v_{\vartheta(j)}^{qn\omega_j}\right)^{k_j}\right)^{1/q}\right)\right\rangle$$

$$\geq\left\langle\left[s_{\prod\limits_{j=1}^{n}\left(n\omega_j\theta^{-}\right)^{k_j}},s_{\prod\limits_{j=1}^{n}\left(n\omega_j\tau^{-}\right)^{k_j}}\right],\left(\prod_{j=1}^{n}\left((1-(1-a^{q})^{n\omega_j})^{1/q}\right)^{k_j},\left(1-\prod_{j=1}^{n}(1-b^{qn\omega_j})^{k_j}\right)^{1/q}\right)\right\rangle.$$

Thus,

$$\left(\frac{1}{n!}\sum_{\vartheta\in S_n}\prod_{j=1}^{n}\left(\mathrm{n}\omega_j\alpha_{\vartheta(j)}\right)^{k_j}\right)^{\frac{1}{\sum_{j=1}^{n}k_j}}$$

$$=\left\langle\left[s_{\left(\frac{1}{n!}\sum\limits_{\vartheta\in S_n}\prod\limits_{j=1}^{n}\left(n\omega_j\theta_{\vartheta(j)}\right)^{k_j}\right)^{\frac{1}{\sum_{j=1}^{n}k_j}}},s_{\left(\frac{1}{n!}\sum\limits_{\vartheta\in S_n}\prod\limits_{j=1}^{n}\left(n\omega_j\tau_{\vartheta(j)}\right)^{k_j}\right)^{\frac{1}{\sum_{j=1}^{n}k_j}}}\right],\right.$$

$$\left(\left(\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\mu_{\vartheta(j)}^{q}\right)^{n\omega_j}\right)^{k_j}\right)\right)^{\frac{1}{n!}}\right)^{1/q}\right)^{\frac{1}{\sum_{j=1}^{n}k_j}},\right.$$

$$\left.\left.\left(1-\left(1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-v_{\vartheta(j)}^{qn\omega_j}\right)^{k_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}k_j}}\right)^{1/q}\right)\right\rangle$$

$$\geq\left\langle\left[s_{\left(\frac{1}{n!}\sum\limits_{\vartheta\in S_n}\prod\limits_{j=1}^{n}\left(n\omega_j\theta^{-}\right)^{k_j}\right)^{\frac{1}{\sum_{j=1}^{n}k_j}}},s_{\left(\frac{1}{n!}\sum\limits_{\vartheta\in S_n}\prod\limits_{j=1}^{n}\left(n\omega_j\tau^{-}\right)^{k_j}\right)^{\frac{1}{\sum_{j=1}^{n}k_j}}}\right],\right.$$

$$\left(\left(\left(1-\left(\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-(1-a^{q})^{n\omega_j})^{k_j}\right)\right)^{\frac{1}{n!}}\right)^{1/q}\right)^{\frac{1}{\sum_{j=1}^{n}k_j}},\right.$$

$$\left.\left.\left(1-\left(1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}(1-b^{qn\omega_j})^{k_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}k_j}}\right)^{1/q}\right)\right\rangle=a^{-}.$$

which means that $\alpha^{-}\leq q-ROULPMM^{K}(\alpha_1,\alpha_2,...,\alpha_n)$. Similarly, we can also prove that $q-ROULPMM^{K}(\alpha_1,\alpha_2,...,\alpha_n)\leq\alpha^{+}$, which completes the proof of Theorem (4).

One of the prominent advantages of q-ROULPMM is that it can capture the various interrelationship between attributes. More Specifically, q-ROULPMM has a parameter vector leading to a flexible aggregation process. By assigning different values to the parameter vector, some special cases can be obtained.

Case1. If $K=(1,0,...,0)$, then q-ROULPMM reduces to

$$q-ROULPMM^{(1,0,\ldots,0)}\left(\alpha_1,\alpha_2,...,\alpha_n\right)=\sum_{j=1}^{n}\left(\left(1+T\left(\alpha_j\right)\right)\alpha_j/\sum_{i=1}^{n}\left(1+T\left(\alpha_i\right)\right)\right), \quad (14)$$

which is the q-rung orhopair uncertain linguistic power averaging operator.

Case2. If $K=(1/n,1/n,...,1/n)$, then q-ROULPMM reduces to

$$q-ROULPMM^{(1/n,1/n,\ldots,1/n)}\left(\alpha_1,\alpha_2,...,\alpha_n\right)=\prod_{j=1}^{n}\alpha_j^{\frac{1+T\left(\alpha_j\right)}{\sum_{i=1}^{n}\left(1+T\left(\alpha_i\right)\right)}} \quad (15)$$

which is q-rung orhopair uncertain linguistic power geometric operator.

Case3. If $K=(1,1,0,...,0)$, then q-ROULPMM reduces to the q-rung orhopair uncertain linguistic fuzzy power Bonferroni mean operator, i.e.

$$\begin{aligned}&q-ROULPMM^{(1,1,0,\ldots,0)}\left(\alpha_1,\alpha_2,...,\alpha_n\right)=\\&\left\langle\left[s_{\left(\frac{1}{n(n-1)}\sum_{j=1,j\neq i}^{n}\omega_j\theta_j\omega_i\theta_i\right)^{\frac{1}{2}}},s_{\left(\frac{1}{n(n-1)}\sum_{j=1,j\neq i}^{n}\omega_j\tau_j\omega_i\tau_i\right)^{\frac{1}{2}}}\right],\right.\\&\left(\left(1-\left(\prod_{\substack{j,i=1\\j\neq i}}^{n}\left(1-\left(1-\left(1-\mu_j^2\right)^{\omega_j}\right)\left(1-\left(1-\mu_i^2\right)^{\omega_i}\right)\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{4}}\right),\\&\left.\left(1-\left(1-\left(\prod_{\substack{j,i=1\\j\neq i}}^{n}\left(v_j^{2\omega_j}+v_i^{2\omega_i}-v_j^{2\omega_j}v_i^{2\omega_i}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}\right)^{1/q}\right\rangle.\end{aligned} \quad (16)$$

Case4. If $K=\left(\overbrace{1,1,...,1}^{k},\overbrace{0,0,...,0}^{n-k}\right)$, then PULPMM reduces to the q-rung othopair uncertain linguistic power Maclaurin symmetric mean operator, i.e

$$\begin{aligned}&q-ROULPMM\left(\overbrace{1,1,...,1}^{k},\overbrace{0,0,...,0}^{n-k}\right)\left(\alpha_1,\alpha_2,...,\alpha_n\right)=\\&\left\langle\left[s_{\left(\frac{\sum_{1\leq j_1<\ldots<j_k\leq n}\prod_{i=1}^{n}\omega_{j_i}\theta_{j_i}}{C_n^k}\right)^{\frac{1}{k}}},s_{\left(\frac{\sum_{1\leq j_1<\ldots<j_k\leq n}\prod_{i=1}^{n}\omega_{j_i}\tau_{j_i}}{C_n^k}\right)^{\frac{1}{k}}}\right],\right.\\&\left(\left(1-\prod_{1\leq j_1<\ldots<j_k\leq n}\left(1-\prod_{i=1}^{k}\left(1-\left(1-\mu_{j_i}^q\right)^{\omega_{j_i}}\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{qk}}\right),\\&\left.\left(1-\left(1-\prod_{1\leq j_1<\ldots<j_k\leq n}\left(1-\prod_{i=1}^{k}\left(1-v_{j_i}^{q\omega_{j_i}}\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}\right)^{1/q}\right\rangle.\end{aligned} \quad (17)$$

3.2 The weighted q-rung othopair uncertain linguistic power Muirhead mean operator

The q-ROULPMM does not consider the importance of the aggregated q-ROULVs. In this subsection, its weighted form, namely, the q-orthopair uncertain linguistic weighted power Muirhead mean (q-ROULWPMM) operator, has been proposed to copy with the weight information of attributes.

Definition 11 Let $\alpha_j = \left\langle \left[s_{\theta_j}, s_{\tau_j}\right], (\mu_j, v_j) \right\rangle (j = 1, 2, ..., n)$ be a collection q-ROULVs and $K = (k_1, k_2, ..., k_n) \in R^n$ be a set of parameters. Then the q-ROULWPMM is defined as follows,

$$q - ROULWPMM^K(\alpha_1, \alpha_2, ..., \alpha_n) = \left(\frac{1}{n!} \sum_{\vartheta \in S_n} \prod_{j=1}^{n} \left(\frac{n\omega_{\vartheta(j)} w_{\vartheta(j)}}{\sum_{i=1}^{n} \omega_i w_i} \alpha_{\vartheta(j)} \right)^{k_j} \right)^{\frac{1}{\sum_{j=1}^{n} k_j}} \tag{18}$$

where $w = (w_1, w_2, ..., w_n)^T$ is the weight vector of $\alpha_j \ (j = 1, 2, ..., n)$ satisfying $w_j \in [0, 1]$ and $\sum\limits_{j=1}^{n} w_j = 1$. $\omega_j = \ (1 + T(\alpha_j)) / \sum_{i=1}^{n} (1 + T(\alpha_i))$ is the PWV, satisfying $\omega_j \in [0, 1]$, $\sum_{j=1}^{n} \omega_j = 1$. $T(\alpha_j) = \sum_{i=1, i \neq j}^{n} Sup(\alpha_i, \alpha_j)$ and $Sup(\alpha_i, \alpha_j)$ represents the support degree between α_i and α_j.

Similarly, we can obtain that the q-ROULWPMM satisfies following property.

Theorem 5 Let $\alpha_j = \left\langle \left[s_{\theta_j}, s_{\tau_j}\right], (\mu_j, v_j) \right\rangle (j = 1, 2, ..., n)$ be a collection q-ROULVs and $K = (k_1, k_2, ..., k_n) \in R^n$ be a set of parameters. Then, the aggregated value by q-ROULWPMM is also a q-ROULV, and

$$\begin{aligned} & q - ROULWPMM^K(\alpha_1, \alpha_2, ..., \alpha_n) \\ & = \left\langle \left[s_{\left(\frac{1}{n!} \sum_{\vartheta \in S_n} \prod_{j=1}^{n} \left(\frac{n\omega_{\vartheta(j)} w_{\vartheta(j)}}{\sum_{i=1}^{n} \omega_i w_i} \theta_{\vartheta(j)} \right)^{k_j} \right)^{\frac{1}{\sum_{j=1}^{n} k_j}}}, s_{\left(\frac{1}{n!} \sum_{\vartheta \in S_n} \prod_{j=1}^{n} \left(\frac{n\omega_{\vartheta(j)} w_{\vartheta(j)}}{\sum_{i=1}^{n} \omega_i w_i} \tau_{\vartheta(j)} \right)^{k_j} \right)^{\frac{1}{\sum_{j=1}^{n} k_j}}} \right], \right. \\ & \left(\left(\left(1 - \left(\prod_{\vartheta \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \left(1 - \mu_{\vartheta(j)}^q \right)^{\frac{n\omega_{\vartheta(j)} w_{\vartheta(j)}}{\sum_{i=1}^{n} \omega_i w_i}} \right)^{k_j} \right) \right)^{\frac{1}{n!}} \right)^{1/q} \right)^{\frac{1}{\sum_{j=1}^{n} k_j}}, \right. \\ & \left. \left. \left(1 - \left(1 - \prod_{\vartheta \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \left(v_{\vartheta(j)} \right)^{\frac{q\omega_{\vartheta(j)} w_{\vartheta(j)}}{\sum_{i=1}^{n} \omega_i w_i}} \right)^{k_j} \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} k_j}} \right) \right) \right\rangle. \end{aligned} \tag{19}$$

The process of proof is similar to that of Theorem (2), so it is omitted here.

4 A novel approach to MAGDM based on the proposed operators

In this section, we introduce a novel approach based on the PMM operators to the MAGDM problems with q-rung othopair uncertain linguistic information. Assume that there are m alternatives $\{x_1, x_2, ..., x_m\}$, $\{G_1, G_2, ..., G_n\}$ be a collection of attributes with the weight vector $w = (w_1, w_2, ..., w_n)^T$, where $w_j \in [0, 1] \ (j = 1, 2..., n)$ and $\sum_{j=1}^{n} w_j = 1$. $D = \{D_1, D_2, ..., D_t\}$ be a collection of decision makers, and their weight vector is $\lambda = (\lambda_1, \lambda_2, ..., \lambda_t)^T$, satisfying $\lambda_p \in [0, 1]$ and $\sum_{p=1}^{t} \lambda_p = 1$. For the attribute $G_{\mathrm{j}} \ (j = 1, 2, ..., n)$ of alternative $x_i \ (i = 1, 2, ..., m)$, decision makers are required to use a q-ROULV $\alpha_{ij} = \left\langle \left[s_{\theta_{ji}}, s_{\tau_{ji}}\right], (\mu_{ji}, v_{ji}) \right\rangle$ to express their preference information. Therefore, a q-rung othopair uncertain linguistic decision matrix can be obtained, which can be denoted as $A^p = \left(\alpha_{ij}^p\right)_{m \times n}$. In the following, we utilize the proposed q-rung othopair uncertain linguistic aggregation operators to solve this problem.

4.1 The model based on cross entropy to obtain the weight vector of attributes

In practical MAGDM problem, the attribute weights is unknown or partly unknown, we need to calculate the attribute weights by using the maximum cross-entropy method. Based on [43], a model based on cross entropy of q-ROULVs is established to obtain the weight vector of attributes.

For the alternatives x_i to all the other alternatives for attribute G_j, it deviation can be firstly calculated by:

$$D_{ij} = \frac{1}{m-1} \sum_{r=1, r \neq i}^{m} CE^* \left(\alpha_{ij}, \alpha_{rj} \right) = \frac{1}{2(m-1)} \sum_{r=1, r \neq i}^{m} \left(CE(\alpha_{ij}, \alpha_{rj}) + CE(\alpha_{rj}, \alpha_{ij}) \right) \tag{20}$$

The overall deviation of all alternatives to other alternatives for attribute G_j is represented as:

$$D_j = \sum_{i=1}^{m} D_{ij} = \frac{1}{m-1} \sum_{i=1}^{m} \sum_{r=1, r \neq i}^{m} CE^*(\alpha_{ij}, \alpha_{rj}) \tag{21}$$

It means that we can obtaining the optimal attributes weights by constructing the a linear programming model

$$(\mathrm{M}-1) \begin{cases} \max D(w) = \sum_{j=1}^{n} w_j D_j = \sum_{j=1}^{n} \sum_{i=1}^{m} D_{ij} w_j \\ \text{Subject to } \sum\limits_{j=1}^{n} w_j = 1, w_j \geq 0, j = 1, 2, \ldots, n \end{cases} \tag{22}$$

Solve the Eq.(22), we can get

$$w_j = \frac{\sum\limits_{i=1}^{m} \left(\frac{1}{2(m-1)} \sum\limits_{r=1, r \neq i}^{m} \left(CE(\alpha_{ij}, \alpha_{rj}) + CE(\alpha_{rj}, \alpha_{ij}) \right) \right)}{\sum\limits_{j=1}^{n} \sum\limits_{i=1}^{m} \left(\frac{1}{2(m-1} \sum\limits_{r=1, r \neq i}^{m} \left(CE(\alpha_{ij}, \alpha_{rj}) + CE(\alpha_{rj}, \alpha_{ij}) \right) \right)} \tag{23}$$

4.2 The Decision making process

Step 1. Normalize the decision matrices, the attributes can be generally divided into two types: benefit attributes and cost attributes. Therefore, the decision matrices should be normalized according to the following formula:

$$\alpha_{ij}^{p} = \begin{cases} \left\langle [s_{\theta_{ij}^{p}}, s_{\tau_{ij}^{p}}], (\mu_{ij}^{p}, v_{ij}^{p}) \right\rangle & G_i \in I_1 \\ \left\langle [s_{\theta_{ij}^{p}}, s_{\tau_{ij}^{p}}], (v_{ij}^{p}, \mu_{ij}^{p}) \right\rangle & G_i \in I_2 \end{cases} \tag{24}$$

where I_1 and I_2 represent benefit type and the cost type attributes respectively.

Step 2. Utilize the q-rung orthopair uncertain linguistic weighted average (q-ROULWA) operator to aggregate the assessments of all $x_i, (i = 1, 2, ..., m)$ by t decision makers with respect to attribute G_j

$$\alpha_{ij} = q - ROULWPMM^{K} \left(\alpha_{ij}^{1}, \alpha_{ij}^{2}, ..., \alpha_{ij}^{t} \right) \tag{25}$$

Step 3. Calculate the weight vector of attributes

$$w_j = \frac{\sum\limits_{i=1}^{m} \left(\frac{1}{2(m-1)} \sum\limits_{r=1, r \neq i}^{m} \left(CE(\alpha_{ij}, \alpha_{rj}) + CE(\alpha_{rj}, \alpha_{ij}) \right) \right)}{\sum\limits_{j=1}^{n} \sum\limits_{i=1}^{m} \left(\frac{1}{2(m-1} \sum\limits_{r=1, r \neq i}^{m} \left(CE(\alpha_{ij}, \alpha_{rj}) + CE(\alpha_{rj}, \alpha_{ij}) \right) \right)} \tag{26}$$

Step 4. Utilize the q-ROULWPMM operator to obtain the aggregation value of x_i with respect to attribute $G_j, (j = 1, 2, ..., n)$

$$\alpha_i = q - ROULWPMM^{K} \left(\alpha_{i1}, \alpha_{i2}, ..., \alpha_{in} \right) \tag{27}$$

Step 5. According to Definition (5), calculate the expected values of the overall preference values $\alpha_i \, (i = 1, 2, ..., m)$

Step 6. Rank alternatives $\{x_1, x_2, ..., x_m\}$ and select the optimal alternative(s).

Table 1: Intuitionistic uncertain linguistic decision matrix R^1

	G_1	G_2	G_3	G_4
A_1	<[s5,s5],(0.2,0.7)>	<[s2,s3],(0.4,0.6)>	<[s5,s6],(0.5,0.5)>	<[s3,s4],(0.2,0.6)>
A_2	<[s4,s5],(0.4,0.6)>	<[s5,s5],(0.4,0.5)>	<[s3,s4],(0.1,0.8)>	<[s4,s4],(0.5,0.5)>
A_3	<[s3,s4],(0.2,0.7)>	<[s4,s4],(0.2,0.7)>	<[s4,s5],(0.3,0.7)>	<[s4,s5],(0.2,0.7)>
A_4	<[s6,s6],(0.5,0.4)>	<[s2,s3],(0.2,0.8)>	<[s3,s4],(0.2,0.6)>	<[s3,s3],(0.3,0.6)>

Table 2: Intuitionistic uncertain linguistic decision matrix R^2

	G_1	G_2	G_3	G_4
A_1	<[s4,s4],(0.1,0.7)>	<[s3,s4],(0.2,0.7)>	<[s3,s4],(0.2,0.8)>	<[s6,s6],(0.4,0.5)>
A_2	<[s5,s6],(0.4,0.5)>	<[s3,s4],(0.3,0.6)>	<[s4,s5],(0.2,0.6)>	<[s3,s4],(0.2,0.7)>
A_3	<[s4,s5],(0.2,0.6)>	<[s4,s4],(0.2,0.7)>	<[s2,s3],(0.4,0.6)>	<[s3,s4],(0.3,0.7)>
A_4	<[s5, s5],(0.3,0.6)>	<[s4,s5],(0.4,0.5)>	<[s2,s3],(0.3,0.6)>	<[s4,s4],(0.2,0.6)>

5 Numerical Example

To illustrate the validity and superiorities of the proposed approach, we provide a numerical example as well as some comparative analysis in this section. This example is adopted from Liu and Jin [28]. An investment company wants to invest its money to a company and after primary evaluation, there are four possible alternatives on the candidates list. They are: (1) A_1 is a care company, (2) A_2 is a computer company, (3) A_3 is a TV company, and (4) A_4 is a food company. In order to select the best alternative, the four companies are evaluated from four attributes: (1) G_1 is the risk analysis, (2) G_2 is the growth analysis, (3) G_3 is the social political impact analysis, and (4) G_4 is the environmental impact analysis. The weight vector of the attributes is $w = (0.32, 0.26, 0.18, 0.24)^T$. The investment company invites three experts to be the decision-making committee. Decision makers whose weight vector is $\lambda = (0.4, 0.32, 0.28)^T$ are required to use the linguistic term set $S = \{s_0, s_1, s_2, s_3, s_4, s_5, s_6\}$ to assess the four alternatives respectively. Therefore, the decision matrices $R^k = \left[\alpha_{ij}^k\right]_{4\times4}$ can be obtained, which are shown in Tables 1-3.

5.1 Decision-making process

In this section, we utilize the proposed method to solve the above problem.

Step 1: Evidently, all the attributes are of the benefit type. Thus, they do not have to be standardized.

Step 2: Utilize the q-ROULWA operator to calculate the collective decision matrix, which is shown as Table 4.

Step 3: Calculate the weight vector of attributes as

$$w^* = (0.2591, 0.2476, 0.2709, 0.1864)$$

Step 4: For alternatives $A_i\,(i = 1, 2, 3, 4)$, utilize the q-ROULWPMM operator to calculate its comprehensive evaluation value.

Table 3: Intuitionistic uncertain linguistic decision matrix R^3

	G_1	G_2	G_3	G_4
A_1	<[s5,s5],(0.2,0.6)>	<[s3,s4],(0.3,0.7)>	<[s4,s5],(0.4,0.5)>	<[s4,s4],(0.2,0.7)>
A_2	<[s4,s5],(0.3,0.7)>	<[s5,s5],(0.3,0.6)>	<[s2,s3],(0.1,0.8)>	<[s3,s4],(0.4,0.6)>
A_3	<[s4,s4],(0.2,0.7)>	<[s5,s5],(0.3,0.6)>	<[s1,s3],(0.1,0.8)>	<[s4,s4],(0.2,0.7)>
A_4	<[s3,s4],(0.2,0.7)>	<[s3,s4],(0.1,0.7)>	<[s4,s5],(0.1,0.7)>	<[s5,s5],(0.4,0.5)>

Table 4: The collective decision matrix

	G_1	G_2
A_1	<[s4.5886,s4.5886],(0.1578,0.6727)>	<[s2.5975,s3.6021],(0.2868,0.6768)>
A_2	<[s4.2503,s5.2411],(0.3606,0.6203)>	<[s4.1717,s4.5916],(0.3280,0.5803)>
A_3	<[s3.5936,s4.2607],(0.1989,0.6743)>	<[s4.2525,s4.2525],(0.2276,0.6730)>
A_4	<[s4.4429,s4.8900],(0.3084,0.6055)>	<[s2.8547,s3.8745],(0.1990,0.6946)>
	G_3	G_4
A_1	<[s3.8689,s4.8745],(0.3390,0.6516)>	<[s4.1073,s4.5207],(0.2506,0.6193)>
A_2	<[s2.8519,s3.8706],(0.1253,0.7528)>	<[s3.2711,s3.9627],(0.3397,0.6205)>
A_3	<[s1.9812,s3.5235],(0.2277,0.7190)>	<[s3.2688,s4.2656],(0.2875,0.6722)>
A_4	<[s2.8479,s3.8652],(0.2607,0.6062)>	<[s3.8720,s3.8720],(0.2871,0.5724)>

Table 5: Ranking results by using the different parameter vector k

K	$E(\alpha_1)$	$E(\alpha_2)$	$E(\alpha_3)$	$E(\alpha_4)$	Ranking results
$k=(1,0,0,0)$	1.3020	1.4550	1.1142	1.3451	$A_2 \succ A_1 \succ A_4 \succ A_3$
$k=(1,1,0,0)$	1.2724	1.3713	1.0749	1.2969	$A_2 \succ A_1 \succ A_4 \succ A_3$
$k=(1,1,1,0)$	1.2498	1.3014	1.0413	1.2631	$A_2 \succ A_1 \succ A_4 \succ A_3$
$k=(1,1,1,1)$	1.2295	1.2156	1.0040	1.2338	$A_1 \succ A_4 \succ A_2 \succ A_3$

$\alpha_1 = \langle [\mathrm{s}_{3.6365}, s_{4.2823}], (0.2469, 0.6633)\rangle$, $\alpha_2 = \langle [s_{3.5044}, s_{4.2831}], (0.2627, 0.6668)\rangle$
$\alpha_3 = \langle [s_{3.0812}, s_{3.9684}], (0.2307, 0.6954)\rangle$, $\alpha_4 = \langle [s_{3.3724}, s_{4.0241}], (0.2577, 0.6318)\rangle$

Step 4: Calculate the expected values $E(\alpha_i)$ of alternative A_i $(i = 1, 2, 3, 4)$, and we can get: $E(\alpha_1) = 1.2295$, $E(\alpha_2) = 1.2156$, $E(\alpha_3) = 1.0040$, $E(\alpha_4) = 1.2338$

Step 5: According to the expected values of alternatives, we can obtain their ranking order, i.e. $A_1 \succ A_4 \succ A_2 \succ A_3$. Hence, A_1 is the best alternative.

5.2 The influence of the parameters on the results

In this subsection, we investigate the influence of the parameter vector k on the decision results. So, we set different parameter vectors k in the q-ROULWPMM operator and discuss the ranking results. The details are in Table 5.

Table 5 shows that when K takes different values, the expected values and ranking orders are also changed relevantly. At the same time, it can be seen that the more relevant ship between attributes is considered, the smaller the expected values will become. Therefore, we can regard the parameter vector K as the decision-maker's attitude toward optimism or pessimism. In this way, decision makers can express their optimistic or pessimistic attitudes and actual needs by changing the parameter vector K.

Table 6: Ranking results obtained by using different methods

Methods	Ranking results
IULWGA operator [37]	$A_2 \succ A_4 \succ A_1 \succ A_3$
IULWGHM operator [38]	$A_2 \succ A_4 \succ A_1 \succ A_3$
IULWBM operator [39]	$A_2 \succ A_4 \succ A_3 \succ A_1$
WPFULMSM operator [40]	$A_2 \succ A_3 \succ A_1 \succ A_4$
q-ROWPULPMM operator	$A_1 \succ A_4 \succ A_2 \succ A_3$

Table 7: Characteristics of different methods

Method	Whether it has felxible power for describing uncertainty	Whether it can discount outliers	Whether it captures the relationship between arguments
IULWGA [37]	No	No	No
IULWGHM [38]	No	No	Yes
IULWBM [39]	No	No	Yes
WPFULMSM [40]	Yes	No	Yes
WPULPMM	Yes	Yes	Yes
Method	**Whether it captures the relationship among arguments**	**Whether it captures the relationship amng all arguments**	**Whether it consider the self-importance of arguments**
IULWGA [37]	No	No	No
IULWGHM [38]	No	No	Yes
IULWBM [39]	No	No	Yes
WPFULMSM [40]	Yes	No	No
WPULPMM	Yes	Yes	Yes

5.3 Comparative analysis

In this section, we compare the q-ROULWPMM operator proposed in this paper with the example which has been mentioned above. (1) The weighted geographic mean operator based on intuitionistic uncertainty (IULWGA) proposed by Liu and Jin [28]; (2) The weighted arithmetic operator based on intuitionistic uncertainty (IULWGHM) proposed by Liu et al.[44]; (3) That introduced by Liu et al.[45] based on learning uncertain language weighted Bonferroni mean (IULWBM) operator; (4) That proposed by Liu et al.[46] based on weighted Pythagorean fuzzy Determine the language Maclaurin Symmetric Mean (WPFULMSM) operator. The decision results of the various operators for the above examples are presented in Table 6. Next, we will conduct a detailed comparative analysis based on this.

First, the methods proposed by Liu and jin [28], Liu et al.[44, 45] are based on IULSs. As we have already mentioned above, q-rung orthopair uncertain linguistic sets (PFULSs) are more powerful than IULSs. For example, if the sum of the membership and non-membership of the ULV provided by the decision maker is greater than 1, like $\langle [s_5, s_6], (0.7, 0.8)\rangle$then we cannot select IULSs to represent the set of fuzzy numbers. Therefore, the method proposed in this paper is more flexible and powerful than other methods.

Liu and Jin's [28] method is based on the IULWGA , which does not consider the interrelationship between attribute values. The method [44, 45] of Liu et al. is based on the IULWAHM and IULWBM, Compared to the IULWGA operator, these two operators take the relationship between the attribute values as a consideration. However, its drawback is that it can only capture the relationship between any two attribute values. This still does not satisfy most of the actual situation. Later, Liu et al. [46] proposed the Maclaurin symmetric mean (MSM) operator, which also demonstrates the correlation between attribute values. Compared with the operators introduced in the previous section, the method proposed by Liu et al. [46] is more practical and powerful. Because the Maclaurin Symmetric Mean (MSM) operator can capture the correlation between two or more attribute values, the larger the k value, the more correlation between the attribute values can be captured. However, the MSM operator also has its drawbacks. The MSM operator can only consider capturing the correlation between n-1 attribute values at most. In addition, the MSM operator cannot reflect the importance of individuals among the aggregated parameters.

The PMM operator proposed in this paper is produced by combining the PA operator with the MM operator. First of all, the PMM operator can well capture the correlation between attribute values. It is worth mentioning that the MSM operator is a special case of the MM operator, and the PMM operator is derived by the MM operator. Therefore, the PMM operator not only considers the correlation between all attribute values, but also lists the individual's level as a consideration. In addition, the PMM operator has a parameter vector, which makes the information aggregation process more flexible and feasible. Secondly, the PMM operator also has the characteristics of the PA operator, which allows the evaluation values to support and strengthen each other, so it can well avoid the situation where the value of the attribute provided by the decision maker is too high or too low. So, in summary, the PMM operator proposed in this paper is more powerful, more flexible, and more versatile.

6 Conclusions

In this paper, we develops a MAGDM method based on q-rung orthopair uncertain PMM operator with unknown attributes weights. In order to do this, we firstly define the cross-entropy of q-ROULSs, which is utilized to obtain the optimal weight vector of attributes by a linear programming model. Secondly, we first propose q-rung orthopair uncertain linguistic PMM (q-ROULPMM) operator and its weighted form to summarize the decision maker's preference information and determine the best choice. Then, based on this, we introduced a new MAGDM method. then, we apply this method to investment project selection issues. Later, in order to better demonstrate the advantages and superiority of the proposed method, we compare it with other methods in terms of qualitative and quantitative. In future work, we are going to apply PMM operators to more fuzzy linguistic environments, such as hesitant fuzzy linguistic sets [47], probabilistic linguistic term sets [48], and so on.

Funding

This work was partially supported by a major project of National Social Science Foundation of China with grant number 18ZDA086.

Author contributions

The idea of the whole thesis was put forward by Hongmei Zhao. She also wrote the paper. Runtong Zhang analyzed the existing work and Xiaomin Zhu provided the numerical instance. The computation of the paper was conducted by Ao Zhang. All authors have read and agreed to the published version of the manuscript

Conflict of interest

The authors declare no conflict of interest.

References

[1] Lotfi A Zadeh. Information and control. *Fuzzy sets*, 8(3):338–353, 1965.

[2] Krassimir Atanassov. Intuitionistic fuzzy sets. *International Journal Bioautomation*, 20:1, 2016.

[3] Ronald R Yager. Pythagorean membership grades in multicriteria decision making. *IEEE Transactions on Fuzzy Systems*, 22(4):958–965, 2013.

[4] Shikha Maheshwari and Amit Srivastava. Study on divergence measures for intuitionistic fuzzy sets and its application in medical diagnosis. *Journal of Applied Analysis & Computation*, 6(3):772–789, 2016.

[5] Mahatab Uddin Molla, Bibhas C Giri, and Pranab Biswas. Extended promethee method with pythagorean fuzzy sets for medical diagnosis problems. *Soft Computing*, pages 1–10, 2021.

[6] Wen-Sheng Chou. New algorithm of similarity measures for pattern-recognition problems. *Journal of Testing and Evaluation*, 44(4):1473–1484, 2016.

[7] Shyi-Ming Chen, Shou-Hsiung Cheng, and Tzu-Chun Lan. A novel similarity measure between intuitionistic fuzzy sets based on the centroid points of transformed fuzzy numbers with applications to pattern recognition. *Information Sciences*, 343:15–40, 2016.

[8] Murat Olgun, Mehmet Ünver, and Şeyhmus Yardımcı. Pythagorean fuzzy points and applications in pattern recognition and pythagorean fuzzy topologies. *Soft Computing*, pages 1–8, 2021.

[9] Zhong Wang, Zeshui Xu, Shousheng Liu, and Zeqing Yao. Direct clustering analysis based on intuitionistic fuzzy implication. *Applied Soft Computing*, 23:1–8, 2014.

[10] Zhong Wang, Zeshui Xu, Shousheng Liu, and Jian Tang. A netting clustering analysis method under intuitionistic fuzzy environment. *Applied Soft Computing*, 11(8):5558–5564, 2011.

[11] Zhenhua Zhang, Yong Hu, Chao Ma, Jinhui Xu, Shenguo Yuan, and Zhao Chen. Incentive-punitive risk function with interval valued intuitionistic fuzzy information for outsourced software project risk assessment. *Journal of Intelligent & Fuzzy Systems*, 32(5):3749–3760, 2017.

[12] Ningxin Xie, Zhaowen Li, and Gangqiang Zhang. An intuitionistic fuzzy soft set method for stochastic decision-making applying prospect theory and grey relational analysis. *Journal of Intelligent & Fuzzy Systems*, 33(1):15–25, 2017.

[13] Xiaomin Zhu, Kaiyuan Bai, Jun Wang, Runtong Zhang, and Yuping Xing. Pythagorean fuzzy interaction power partitioned bonferroni means with applications to multi-attribute group decision making. *Journal of Intelligent & Fuzzy Systems*, 36(4):3423–3438, 2019.

[14] Liguo Fei and Yong Deng. Multi-criteria decision making in pythagorean fuzzy environment. *Applied Intelligence*, 50(2):537–561, 2020.

[15] Ronald R Yager. Generalized orthopair fuzzy sets. *IEEE Transactions on Fuzzy Systems*, 25(5):1222–1230, 2016.

[16] Peide Liu and Peng Wang. Some q-rung orthopair fuzzy aggregation operators and their applications to multiple-attribute decision making. *International Journal of Intelligent Systems*, 33(2):259–280, 2018.

[17] Peide Liu and Peng Wang. Multiple-attribute decision-making based on archimedean bonferroni operators of q-rung orthopair fuzzy numbers. *IEEE Transactions on Fuzzy systems*, 27(5):834–848, 2018.

[18] Adjei Peter Darko and Decui Liang. Some q-rung orthopair fuzzy hamacher aggregation operators and their application to multiple attribute group decision making with modified edas method. *Engineering Applications of Artificial Intelligence*, 87:103259, 2020.

[19] Wen Sheng Du. Minkowski-type distance measures for generalized orthopair fuzzy sets. *International Journal of Intelligent Systems*, 33(4):802–817, 2018.

[20] Decui Liang, Yinrunjie Zhang, and Wen Cao. q-rung orthopair fuzzy choquet integral aggregation and its application in heterogeneous multicriteria two-sided matching decision making. *International Journal of Intelligent Systems*, 34(12):3275–3301, 2019.

[21] Liuxin Chen, Nanfang Luo, and Xiaoling Gou. A novel q-rung orthopair fuzzy todim approach for multi-criteria group decision making based on shapley value and relative entropy. *Journal of Intelligent & Fuzzy Systems*, (Preprint):1–16.

[22] Adem Pınar, Rouyendegh Babak Daneshvar, and Yavuz Selim Özdemir. q-rung orthopair fuzzy topsis method for green supplier selection problem. *Sustainability*, 13(2):985, 2021.

[23] Jia-Wei Gong, Qiang Li, Linsen Yin, and Hu-Chen Liu. Undergraduate teaching audit and evaluation using an extended mabac method under q-rung orthopair fuzzy environment. *International Journal of Intelligent Systems*, 35(12):1912–1933, 2020.

[24] Xindong Peng and Zhigang Luo. A review of q-rung orthopair fuzzy information: bibliometrics and future directions. *Artificial Intelligence Review*, pages 1–70, 2021.

[25] Kaiyuan Bai, Xiaomin Zhu, Jun Wang, and Runtong Zhang. Some partitioned maclaurin symmetric mean based on q-rung orthopair fuzzy information for dealing with multi-attribute group decision making. *Symmetry*, 10(9):383, 2018.

[26] Dejian Yu, Deng-Feng Li, Jose M Merigo, and Lincong Fang. Mapping development of linguistic decision making studies. *Journal of Intelligent & Fuzzy Systems*, 30(5):2727–2736, 2016.

[27] Zeshui Xu. Uncertain linguistic aggregation operators based approach to multiple attribute group decision making under uncertain linguistic environment. *Information sciences*, 168(1-4):171–184, 2004.

[28] Peide Liu and Fang Jin. Methods for aggregating intuitionistic uncertain linguistic variables and their application to group decision making. *Information Sciences*, 205:58–71, 2012.

[29] Peide Liu. Some geometric aggregation operators based on interval intuitionistic uncertain linguistic variables and their application to group decision making. *Applied Mathematical Modelling*, 37(4):2430–2444, 2013.

[30] Yushui Geng, Peide Liu, Fei Teng, and Zhengmin Liu. Pythagorean fuzzy uncertain linguistic todim method and their application to multiple criteria group decision making. *Journal of Intelligent & Fuzzy Systems*, 33(6):3383–3395, 2017.

[31] Kaiyuan Bai, Xiaomin Zhu, Jun Wang, and Runtong Zhang. Power partitioned heronian mean operators for q-rung orthopair uncertain linguistic sets with their application to multiattribute group decision making. *International Journal of Intelligent Systems*, 35(1):3–37, 2020.

[32] Jun Wang, Runtong Zhang, Li Li, Xiaomin Zhu, and Xiaopu Shang. A novel approach to multi-attribute group decision making based on q-rung orthopair uncertain linguistic information. *Journal of Intelligent & Fuzzy Systems*, 36(6):5565–5581, 2019.

[33] Ronald R Yager. The power average operator. *IEEE Transactions on Systems, Man, and Cybernetics-Part A: Systems and Humans*, 31(6):724–731, 2001.

[34] Carlo Bonferroni. Sulle medie multiple di potenze. *Bollettino dell'Unione Matematica Italiana*, 5(3-4):267–270, 1950.

[35] Stanislav Sỳkora. Mathematical means and averages: Generalized heronian means. *Stan's Library: Castano Primo, Italy*, 2009.

[36] Robert Franklin Muirhead. Some methods applicable to identities and inequalities of symmetric algebraic functions of n letters. *Proceedings of the Edinburgh Mathematical Society*, 21:144–162, 1902.

[37] Colin Maclaurin. A second letter to martin folkes, esq.; concerning the roots of equations, with demonstration of other rules of algebra. *Philos Trans Roy Soc London Ser A*, 36(1729):59–96, 1729.

[38] Li Li, Runtong Zhang, Jun Wang, Xiaomin Zhu, and Yuping Xing. Pythagorean fuzzy power muirhead mean operators with their application to multi-attribute decision making. *Journal of Intelligent & Fuzzy Systems*, 35(2):2035–2050, 2018.

[39] Wuhuan Xu, Xiaopu Shang, Jun Wang, and Weizi Li. A novel approach to multi-attribute group decision-making based on interval-valued intuitionistic fuzzy power muirhead mean. *Symmetry*, 11(3):441, 2019.

[40] Peide Liu, Qaisar Khan, and Tahir Mahmood. Some single-valued neutrosophic power muirhead mean operators and their application to group decision making. *Journal of Intelligent & Fuzzy Systems*, 37(2):2515–2537, 2019.

[41] Zhengmin Liu, Hongxue Xu, Yuannian Yu, and Junqing Li. Some q-rung orthopair uncertain linguistic aggregation operators and their application to multiple attribute group decision making. *International Journal of Intelligent Systems*, 34(10):2521–2555, 2019.

[42] Zhengmin Liu, Lin Li, and Junqing Li. q-rung orthopair uncertain linguistic partitioned bonferroni mean operators and its application to multiple attribute decision-making method. *International Journal of Intelligent Systems*, 34(10):2490–2520, 2019.

[43] Xiaowen Qi, Changyong Liang, and Junling Zhang. Generalized cross-entropy based group decision making with unknown expert and attribute weights under interval-valued intuitionistic fuzzy environment. *Computers & Industrial Engineering*, 79:52–64, 2015.

[44] Peide Liu, Zhengmin Liu, and Xin Zhang. Some intuitionistic uncertain linguistic heronian mean operators and their application to group decision making. *Applied Mathematics and Computation*, 230:570–586, 2014.

[45] Peide Liu, Yubao Chen, and Yanchang Chu. Intuitionistic uncertain linguistic weighted bonferroni owa operator and its application to multiple attribute decision making. *Cybernetics and Systems*, 45(5):418–438, 2014.

[46] Chao Liu, Guolin Tang, and Peide Liu. An approach to multicriteria group decision-making with unknown weight information based on pythagorean fuzzy uncertain linguistic aggregation operators. *Mathematical problems in Engineering*, 2017, 2017.

[47] Rosa M Rodriguez, Luis Martinez, and Francisco Herrera. Hesitant fuzzy linguistic term sets for decision making. *IEEE Transactions on fuzzy systems*, 20(1):109–119, 2011.

[48] Qi Pang, Hai Wang, and Zeshui Xu. Probabilistic linguistic term sets in multi-attribute group decision making. *Information Sciences*, 369:128–143, 2016.

Cite this paper as:
Zhao H.M., Zhang R.T., Zhang A., Zhu X.M., Zhao H., Zhang R., Zhang A., Zhu X. (2021). Multi-attribute Group Decision Making Method with Unknown Attribute Weights Based on the Q-rung Orthopair Uncertain Linguistic Power Muirhead Mean Operators, *International Journal of Computers Communications & Control*, 16(3), 4214, 2021.
https://doi.org/10.15837/ijccc.2021.3.4214