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On approaching full fuzzy data envelopment analysis and its validation

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Abstract

We approach the full fuzzy data envelopment analysis (DEA) strictly relying on the extension principle. So far in the literature, fuzzy DEA (that uses fuzzy inputs and outputs but crisp weights) and full fuzzy DEA (that uses fuzzy inputs, outputs and weights) were treated distinctly. However, the crisp weights from fuzzy models only act as crisp weights within optimization, but in fact their feasible values finally describe fuzzy weights. As a consequence, the distinction between full fuzzy models and fuzzy models is due to the distinction between establishing or not an a priori shape for the fuzzy weights. In this paper we advance the idea that the methodologies introduced for fuzzy DEA are valuable for full fuzzy DEA as well; and propose a Monte Carlo simulation algorithm to offer an empirical visualization of the shapes of the fuzzy efficiencies of DMUs in full fuzzy DEA. Such visualization firstly can certify whether a solution approach to a full fuzzy DEA derives solutions complying to the extension principle or not; and secondly discloses the fuzzy shapes of the weights obtained by applying a methodology from fuzzy DEA to solving full fuzzy DEA. The complexity of the proposed algorithm is the same as the solution approach to the crisp DEA model that corresponds to the observed full fuzzy DEA model. We report the numerical results of our experiments, compare them to results found in the recent literature, and discuss the misleading consequences of ignoring the extension principle in the context of full fuzzy DEA.

Keywords: data envelopment analysis, efficiency, fuzzy programming.

1 Introduction

Data Envelopment Analysis (DEA) is an operations research tool able to measure the relative efficiencies of a set of Decision-Making Units (DMUs). DEA was initially developed by Charnes et al. [1] and since then was under a continuous spreading. Nowadays DEA continues to have a wide range of applications (see for instance [11], [12] and [14]).

As emphasized in [17], to deal efficiently with common operations research problems in fuzzy environment, more effort should be put in providing a consistent fuzzy operational laws. Emrouznejad and Tavana [5] is a good reference on the performance measurement with fuzzy DEA. More recently, Zhou and Xu [25] presented a wide overview of the researches on fuzzy DEA providing directions for further investigations on its theory development and practical applications.

Full fuzzy optimization models are wider used in the recent literature. For instance, Karimi et al. [9] formulated a full fuzzy linear programming model to determine the needed weights of the criteria, and the scores of the alternatives for a multi-attribute decision-making problem.

Whenever a crisp optimization problem is generalized to its analogue in a fuzzy environment, then the generalization of its solution concept should follow the extension principle in order to provide a relevant analogue solution concept in fuzzy environment. Recent studies on optimization in fuzzy environment with respect to the extension principle are available.

Diniz et al. [4] discussed the optimization of a fuzzy-valued function using Zadeh's extension principle. The objective function was a Zadeh's extension of a function with respect to a parameter and an independent variable. Kupka [10] introduced some results on the approximation of Zadeh's extension of a given function, and studied the quality of the approximation with respect to the choice of the metric on the space of the fuzzy sets. Stanojević and Stanojević [19] discussed the solution concept to transportation problems in intuitionistic fuzzy environment.

A methodology based on the extension principle to fuzzy DEA was proposed by Kao and Liu [7]. With the help of α-cuts of the coefficients, the fuzzy DEA was transformed into a family of crisp DEA models that were further properly linearized and solved. Kao and Liu [8] also used the same idea to derive efficiencies to two-stage systems with fuzzy data. Recently, Soltanzadeh and Omrani [16] applied the same methodology to handle the fuzziness in a dynamic network DEA.

Sotoudeh-Anvari et al. [18] proposed a DEA in fully fuzzy environment on the base of the degree of certainty of information. They used z-numbers to describe the fuzzy uncertainty. Namakin et al. [13] proposed a new approach to fuzzy DEA and used the results provided in [18] for comparison.

In this paper we propose a simple and general Monte Carlo simulation-based algorithm that offers a visualization of the shapes of the fuzzy efficiencies and weights of DMUs, and can be used to certify whether a solution approach to full fuzzy DEA derives realistic solutions (i.e. with respect to the extension principle) or not. The complexity of the proposed algorithm is the same as the solution approach to the crisp DEA models that correspond to the full fuzzy DEA models under validation. This algorithm improves the algorithm that was used in [20] for validating approaches to full fuzzy linear programming problems, since for each parameters it involves exclusively its either lower or upper bound instead of uniformly generating values between its lower and upper bounds.

We illustrate our results using numerical examples from the recent literature [18], [13] and [16]. For our experiments we developed several libraries that facilitate the use of an open source solver in Lua. Having proper modeling and model manipulation capabilities, these libraries provide high computation performances.

Our main goals are: (i) to provide quadratic optimization models able to derive exact solutions to small-size problems; (ii) to propose linear models whose solutions disclose relevant information about the efficiencies of DMUs to large-scale problems, and in full accordance to the extension principle; and overall (iii) to rehabilitate the use of the extension principle within fuzzy DEA.

The rest of the paper is organized as follows. Section 2 includes notation and terminology; basic formulations for crisp, fuzzy and full fuzzy DEA models; and two re-formalized crisp mathematical models for approaching the full fuzzy DEA in accordance to the extension principle. Section 3 contains our new developed algorithms: the first one simulates the extension principle applied to defining the fuzzy efficiencies of DMUs in full fuzzy DEA; and the second one derives the ranking matrix of DMUs based on the comparison of the fuzzy efficiencies in full fuzzy DEA. Section 4 reports our numerical results that illustrate the usefulness and effectiveness of the proposed algorithms. Section 5 includes our conclusions and ideas for a future research.

2 Preliminaries

2.1 Fuzzy sets and fuzzy numbers

Zadeh [23] introduced the concept of fuzzy set $\widetilde{A}$ over the universe X as a collection of pairs $\left(x, \mu_{\widetilde{A}}(x)\right)$, where $x \in X$ and $\mu_{\widetilde{A}}(x) \in [0,1]$. Function $\mu_{\widetilde{A}} : X \to [0,1]$ is the membership function that describes the fuzzy set $\widetilde{A}$, and $\mu_{\widetilde{A}}(x)$ is the membership degree of x in $\widetilde{A}$.

The support of a fuzzy set $\widetilde{A}$ is the set of values with non-zero membership degree, i.e.

$$\text{Supp}\left(\widetilde{A}\right) = \left\{x \in X | \mu_{\widetilde{A}}(x) > 0\right\}.$$

The α-cut of a fuzzy set $\widetilde{A}$ is denoted by $\left[\widetilde{A}\right]_\alpha$ and it is defined as the set of values with a membership degree greater or equal to α, $\alpha > 0$, i.e.

$$\left[\widetilde{A}\right]_\alpha = \left\{x \in X | \mu_{\widetilde{A}}(x) \geq \alpha\right\}.$$

Fuzzy numbers are especial cases of fuzzy sets. A fuzzy set $\widetilde{A}$ of the universe R of real numbers is called fuzzy number if and only if: (i) it is fuzzy normal and fuzzy convex; (ii) its membership function $\mu_{\widetilde{A}}$ is upper semi-continuous; and (iii) its support $\left\{x \in R | \mu_{\widetilde{A}}(x) > 0\right\}$ is bounded. For more details please refer to [3]. The α-cuts of the fuzzy numbers are always intervals.

We use trapezoidal fuzzy numbers in our theoretical presentation. The non-zero piece of the membership function of a trapezoidal fuzzy number forms a trapezoid with the abscissa when it is graphically represented. Generally, a trapezoidal fuzzy number $\widetilde{A}$ is expressed by a quadruple (a_1, a_2, a_3, a_4), $a_1 \leq a_2 \leq a_3 \leq a_4$. The interval (a_1, a_4) is the support of $\widetilde{A}$, and the interval $[a_2, a_3]$ describes the set of values with the membership degree equal to 1.

The α-cut of a trapezoidal fuzzy number $\widetilde{A} = (a_1, a_2, a_3, a_4)$ is the interval

$$\left[\widetilde{A}\right]_\alpha = \left[(1-\alpha)\, a_1 + \alpha a_2, \alpha a_3 + (1-\alpha)\, a_4\right].$$

The inequality $\mu_{\widetilde{A}}(x) > \alpha$ is then equivalent to the double inequality

$$(1-\alpha)\, a_1 + \alpha a_2 \leq x \leq \alpha a_3 + (1-\alpha)\, a_4.$$

Whenever $a_2 = a_3$ in a trapezoidal fuzzy number, that fuzzy number is reduced to a triangular fuzzy number.

2.2 The extension principle

Zadeh [24] proposed the extension principle – as one of the basic ideas in fuzzy set theory – and applied it to develop the fuzzy arithmetic on fuzzy quantities. Ross [15] presented several methods to convert the extended fuzzy operations into efficient computational algorithms.

According to the extension principle, the fuzzy set $\widetilde{B}$ over the universe Y that is the result of evaluating the function f at the fuzzy sets $\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_r$ over their universes $X_1, X_2, \ldots, X_r$ is defined through its membership function

$$\mu_{\widetilde{B}}(y) = \begin{cases} \sup\limits_{(x_1,\ldots,x_r)\in f^{-1}(y)} \left(\min\left\{\mu_{\widetilde{A}_1}(x_1), \ldots, \mu_{\widetilde{A}_r}(x_r)\right\}\right), & f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

See Ross [15] for more details on fuzzy arithmetic and extension principle.

2.3 DEA, fuzzy DEA and full fuzzy DEA models

Let us assume that there are n decision-making units (DMUs) involved in the evaluation process. For each $j = 1, ..., n$ let x_{ij} be the i-th input for the DMU$_j$, $i = 1, ..., m$; and let y_{rj} be the r-th output for the DMU$_j$, $r = 1, ..., s$.

Charnes et al. [1] proposed the classic CCR (Charnes-Cooper-Rhodes) DEA model where the efficiency of any DMU is equal the the maximum of a ratio of weighted outputs to weighted inputs subject to the constraint that for every DMU its corresponding ratio is less than or equal to unity.

A general crisp mathematical model that describes the efficiency θ_p^* of the DMU$_p$, $p = 1, ..., n$ is given in (2).

$$\begin{aligned} \theta_p^* = \max \quad & E_p(u,v) \\ \text{s.t.} \quad & \\ & E_j(u,v) \le 1, \quad j = \overline{1,n}, \\ & (u,v) \in G_{x,y,c}, \\ & u, v \ge \varepsilon, \end{aligned} \tag{2}$$

where $E_j(u,v) = \left(\sum_{r=1}^{s} u_r y_{rj}\right) / \left(\sum_{i=1}^{m} v_i x_{ij}\right)$ represent the efficiency of DMU$_j$ defined with respect to the weights of the inputs and outputs u and v respectively, that have to be determined. The first set of constraints assures that the efficiencies of all DMUs are less or equal to 1. The set $G_{x,y,c}$ contains additional constraints on the weights of the inputs and outputs specific to extended DEA models that contain extra parameters c (such constraints do not appear in the classic CCR model). ε is a small quantity (e.g. $\varepsilon = 10^{-12}$) assuring that all inputs and outputs are considered in the evaluation even with a minor weight.

Assuming that $G_{x,y,c}$ is described by linear constraints, Model (2) can be successfully linearized by the well-known Charnes-Cooper transformation [2] applied to the objective function $E_p(u,v)$; and multiplication of each inequality $E_j(u,v) \le 1$ ($j = \overline{1,n}$) by its strict positive denominator.

Let us denote by $F^{\varepsilon}_{x,y,c}$ the feasible set of the linearized problem (2), and by $H^{\varepsilon}_{x,y,c}$ the feasible set of the dual of the linearized problem (2). If w are the dual variables and $D_p(w)$ is the dual objective function, then

$$\min_{w \in H^{\varepsilon}_{x,y,c}} D_p(w) = \max_{(u,v) \in F^{\varepsilon}_{x,y,c}} E_p(u,v).$$

Fuzzy DEA models have fuzzy coefficients (i.e. inputs and outputs expressed by fuzzy quantities) and crisp variables (i.e. crisp weights). Full fuzzy DEA (FF-DEA) models have both coefficients and variables expressed by fuzzy quantities. However, even for models declared "fuzzy" but not "full fuzzy" there is a hidden "fuzziness" of the variables. To solve a fuzzy or full fuzzy DEA model one must specify how the fuzzy quantities are aggregated, and implicated in optimization.

The solution approaches to fuzzy DEA that essentially follow the extension principle work with crisp variables in their models; while the approaches to full fuzzy DEA a priori impose fuzzy shapes to the variables, but use only their components (endpoints of the support and the values with maximal amplitude) within the optimization models.

The full fuzzy DEA model (3) that corresponds to (2) provides the fuzzy efficiencies $\widetilde{\theta}_p^*$, $p = 1, \ldots, n$ as follows.

$$\begin{aligned} \widetilde{\theta}_p^* = \max \quad & \widetilde{E}_p(\widetilde{u},\widetilde{v}) \\ \text{s.t.} \quad & \\ & \widetilde{E}_j(\widetilde{u},\widetilde{v}) \le \widetilde{1}, \quad j = \overline{1,n}, \\ & (\widetilde{u},\widetilde{v}) \in \widetilde{G}_{\widetilde{x},\widetilde{y},\widetilde{c}}, \\ & \widetilde{u}, \widetilde{v} \ge \varepsilon. \end{aligned} \tag{3}$$

The extension principle given in (1), and adapted to define the membership function $\mu_{\widetilde{\theta}_p^*}$ of the fuzzy efficiency of the DMU$_p$ described in Model (3) is given in (4) at an arbitrary value $z \in [0,1]$; for a fixed index $p \in \{1, \ldots, n\}$; and a small ε, $\varepsilon > 0$.

$$\mu_{\widetilde{\theta}_p^*}(z) = \begin{cases} \underbrace{\max}_{(x,y,c)\,|z = \max\limits_{(u,v)\in F^{\varepsilon}_{x,y,c}} E_p(u,v)} \left(\mu_{\widetilde{x},\widetilde{y},\widetilde{c}}(x,y,c)\right), & \overbrace{z = \max\limits_{(u,v)\in F^{\varepsilon}_{x,y,c}} E_p(u,v)}^{\exists (x,y,c)\,|}, \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

The definition given in (4) is formalized with the help of the crisp Problem (2), and the value $\mu_{\widetilde{x},\widetilde{y},\widetilde{c}}(x, y, c)$ which represents the membership degree of the crisp matrix coefficients x, y and c whose components belong to the support of the components of the fuzzy matrises $\widetilde{x}$, $\widetilde{y}$ and $\widetilde{c}$ respectively.

The value $\mu_{\widetilde{x},\widetilde{y},\widetilde{c}}(x, y, c)$ is computed as

$$\mu_{\widetilde{x},\widetilde{y},\widetilde{c}}(x,y,c) = \min\left\{\mu_{\widetilde{x}}(x), \mu_{\widetilde{y}}(y), \mu_{\widetilde{c}}(c)\right\}. \tag{5}$$

2.4 Crisp mathematical models to approach the full fuzzy DEA

Kao and Liu [7] proposed a family of mathematical models able to describe the left and right endpoints of the α-cut intervals of the fuzzy efficiency $\widetilde{\theta}_p^*$ of DMU_p in the fuzzy DEA model that extends the classic CCR model to fuzzy environment. Their idea can be easily extended to the more general full fuzzy DEA (3). Soltanzadeh and Omrani [16] used it to their fuzzy DEA applied to Iranian Airlines. In what follows we formalize the idea in a more synthetic form, and then show that it can be applied to full fuzzy DEA models as well.

The right endpoint of $\left[\widetilde{\theta}_p^*\right]_\alpha$, denoted by $E_p^U(\alpha)$, is the maximal value z such that there exists the crisp input and output matrices x and y with $\mu_{\widetilde{x},\widetilde{y}}(x, y) \geq \alpha$ that provide the optimal value equal to z to Problem (3). It means that

$$E_p^U(\alpha) = \max_{x,y,c}\left\{\max_{u,v}\left\{E_p(u,v)\,|(u,v)\in F^{\varepsilon}_{x,y,c}\right\}|\mu_{\widetilde{x},\widetilde{y},\widetilde{c}}(x,y,c) \geq \alpha\right\}.$$

Compressing the two maximization steps into a single one the family of models that provide the upper bounds is

$$E_p^U(\alpha) = \max_{x,y,c,u,v}\left\{E_p(u,v)\,|(u,v)\in F^{\varepsilon}_{x,y,c}, \mu_{\widetilde{x},\widetilde{y},\widetilde{c}}(x,y,c) \geq \alpha\right\}. \tag{6}$$

Similarly, the left endpoint of $\left[\widetilde{\theta}_p^*\right]_\alpha$, denoted by $E_p^L(\alpha)$, is the minimal value z such that there exists the crisp input and output matrices x and y with $\mu_{\widetilde{x},\widetilde{y},\widetilde{c}}(x, y, c) \geq \alpha$ that provide the optimal value equal to z to Problem (3). It means that

$$E_p^L(\alpha) = \min_{x,y,c}\left\{\max_{u,v}\left\{E_p(x,y)\,|(u,v)\in F^{\varepsilon}_{x,y,c}\right\}|\mu_{\widetilde{x},\widetilde{y},\widetilde{c}}(x,y,c) \geq \alpha\right\}.$$

To compress the two optimization steps into a single one, the dual was involved, and combined with the minimization over (x, y). It means that the left endpoint of $\left[\widetilde{\theta}_p^*\right]_\alpha$ can be obtained as

$$E_p^L(\alpha) = \min_{x,y,c,w}\left\{D_p(w)\,|w\in H^{\varepsilon}_{x,y,c}, \mu_{\widetilde{x},\widetilde{y},\widetilde{c}}(x,y,c) \geq \alpha\right\}. \tag{7}$$

Both models (6) and (7) are non-linear and it is generally challenging to solved them. Kao and Liu [7] proposed a linearization based on the lower and upper bounds of the α-cuts of inputs and outputs for the specific case of a fuzzy DEA that extends the classic CCR DEA. They successfully replaced the variables x and y by their bounds

$$\begin{aligned} x_p &= (x_p)_\alpha^L, \quad y_p = (y_p)_\alpha^U, \\ x_j &= (x_j)_\alpha^U, \quad y_j = (y_j)_\alpha^L, \quad j = \overline{1,n}, j \neq p. \end{aligned}$$

Soltanzadeh and Omrani [16] applied the same transformations on the inputs and outputs within their corresponding Models (6) and (7), and kept the additional variables c within their lower and upper bounds dictated by the corresponding α-cuts. They also made use of a model-related linearization, thus succeeding to completely linearize their corresponding Model (6). Their corresponding Model (7) remained non-linear. Our second numerical example given in the section of the computational results indicates that their linearized model is not equivalent to the non-linear one, therefore higher efficiencies might be obtained as a result of solving the non-linear Model (6).

In the next section we show how Models (6) and (7) can be used to derive approximate membership functions of the fuzzy efficiencies of DMUs in a full fuzzy DEA, and rank the DMUs in accordance to the extension principle.

3 Our solution algorithms

3.1 The Monte Carlo simulation-based algorithm

In this section we present a Monte Carlo simulation-based algorithm that is able to derive empirically the shapes of the fuzzy number values of DMUs efficiencies in a strict accordance to the extension principle adapted to the FF-DEA model (3).

To simulate the definition of the membership function of the fuzzy efficiency of the DMU_p given in (4), we should randomly generate crisp values with non-zero membership degrees for all DMUs' inputs and outputs, and then use them to solve crisp Problems (2).

According to [7] when the inputs and outputs of every DMU vary in ranges, to find the smallest (highest) relative efficiency of a DMU compared with other DMUs, one will set the output level of this DMU and the input levels of all other DMUs to their lowest (upper) values, and set the input level of this DMU and the output levels of all other DMUs to their upper (lowest) values. In a fuzzified CCR DEA, as the one addressed in [7], the inputs and outputs of DMUs are the only fuzzy parameters involved. However, for more complex fuzzified DEA models, as Model (3), additional fuzzy parameters $\widetilde{c}$ might be involved to describe the set $G_{\widetilde{x},\widetilde{y},\widetilde{c}}$.

Algorithm 1 derives n lists of efficiency values paired with their corresponding membership degrees, one list for each DMU; and $n\,(s+m)$ lists of weight values paired again with their corresponding membership degrees (for each DMU s lists of input weights and m of output weights). The graphic representation of the pairs belonging to the p-th list of efficiencies discloses the membership function of the fuzzy efficiency of the DMU_p, $p = 1, \ldots, n$. Similar, the lists of weights belonging to DMU_p disclose the fuzzy weights of its inputs and outputs.

The algorithm solves crisp DEA problems, and constructs fuzzy efficiencies of DMUs in the corresponding extended fuzzy DEA. The algorithm functions well as long as a convenient algorithm to solve the crisp DEA exists. This is not the case of the solution approach based on Models (6) and (7), since due to their non-linearity these models might become too complex much faster than the corresponding crisp DEA model.

For our first numerical example we use this algorithm to certify that neither Sotoudeh-Anvari et al. [18] nor Namakin et al. [13] derived the fuzzy efficiencies in a full accordance to the extension principle.

In the second example, our Monte Carlo simulation-based numerical results (1) visually confirm that the methodology used by Soltanzadeh and Omrani [16] followed the extension principle until the linearization step, but not further; and (2) depict the fuzzy shapes of the weights of inputs and outputs for the best ranked DMU, thus illustrating our statement that fuzzy DEA works with crisp weights but the derived efficiencies are sustained by fuzzy weights.

The algorithm can be also used to provide useful information on the quality of the results derived by any solution approach to FF-DEA whenever solving Models (6) and (7) becomes burdensome because of either the complexity of the original problem or a numerical instability.

3.2 The algorithm for ranking DMUs

Our procedure that derives a ranking matrix of DMUs in a general full fuzzy DEA is described by Algorithm 2 with the help of Algorithms 3, 4 and 5.

Algorithm 1 The Monte Carlo simulation-based algorithm for FF-DEA

Input: A natural number $q \in N$; two sequences $\alpha_1, \alpha_2, \ldots, \alpha_q \in [0,1]$ for α-cuts, and $\gamma_1, \gamma_2, \ldots, \gamma_q \in N$ for the number of iterations within one α-cut; the membership functions of the fuzzy sets of DMUs' inputs and outputs $\widetilde{x}_{ij}$, $\widetilde{y}_{rj}$, $j = 1, \ldots, n$, $i = 1, \ldots, m$, $r = 1, \ldots, s$.

Set $L_p = \varnothing$, $p = 1, \ldots, n$ and $\varepsilon = 10^{-12}$.
Set $U_{rj} = \varnothing$, and $V_{ij} = \varnothing$, $j = 1, \ldots, n$; $i = 1, \ldots, m$; $r = 1, \ldots, s$.
for $k = \overline{1, q}$ **do**
  for $l = \overline{1, \gamma_k}$ **do**
    Randomly generate $x_{ij} \in [\widetilde{x}_{ij}]_{\alpha_k}$ and $y_{rj} \in [\widetilde{y}_{rj}]_{\alpha_k}$,
    $j = 1, \ldots, n$, $i = 1, \ldots, m$, $r = 1, \ldots, s$, and $c \in [\widetilde{c}]_{\alpha_k}$.
    Compute $\alpha^* = \mu_{\widetilde{x},\widetilde{y},\widetilde{c}}(x, y, c)$ using (5).
    for $p = \overline{1, n}$ **do**
      Solve Problem (2), set θ_p^* as its optimal value, and u_r^* and $v_i{}^*$
      as the optimal weights of DMU_p, $i = 1, \ldots, m$; $r = 1, \ldots, s$.
      Set $L_p = L_p \cup \left\{ \left(\theta_p^*, \alpha^* \right) \right\}$.
      Set $U_{rp} = U_{rp} \cup \{(u_r^*, \alpha^*)\}$, $r = 1, \ldots, s$.
      Set $V_{ip} = V_{ip} \cup \{(v_i^*, \alpha^*)\}$, $i = 1, \ldots, m$.
    end for
  end for
end for

Output: The lists L_j, U_{rj}, V_{ij}, $j = 1, \ldots, n$; $i = 1, \ldots, m$; $r = 1, \ldots, s$ containing DMUs' efficiencies and weights paired with their corresponding membership degrees.

Algorithm 2 to derive the ranking matrix of DMUs

Input: A natural number $q \in N$; a sequence $0 < \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_q \leq 1$; the membership functions of the fuzzy sets of DMUs' inputs and outputs $\widetilde{x}_{ij}$, $\widetilde{y}_{rj}$, $j = 1, \ldots, n$, $i = 1, \ldots, m$, $r = 1, \ldots, s$.

for $j = \overline{1, n}$ **do**
  $f_{jj} = 1$.
  Run Algorithm 3 for $p = j$ and obtain L_j^l and L_j^r.
  Make conclusions about the efficiency or non-efficiency of the DMU_j.
end for
for $j_1 = \overline{1, n}$ **do**
  for $j_2 = \overline{j_1 + 1, n}$ **do**
    Run Algorithm 4 on inputs $L_{j_1}^l$, $L_{j_1}^r$, $L_{j_2}^l$ and $L_{j_2}^r$
    and obtain the lists $L_{j_1 j_2}^l$ and $L_{j_1 j_2}^r$.
    Run Algorithm 5 on inputs $L_{j_1 j_2}^l$ and $L_{j_1 j_2}^r$
    and obtain the output $f_{j_1 j_2}$.
    Define $f_{j_2 j_1} = 1 - f_{j_1 j_2}$.
  end for
end for

Output: The ranking matrix $F = (f_{j_1 j_2})_{j_1 = \overline{1,n}}^{j_2 = \overline{1,n}}$.

Algorithm 3 to derive the fuzzy efficiency of DMU_p

Input: A natural number $q \in N$; a sequence $0 < \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_q \leq 1$; the membership functions of the fuzzy sets of DMUs' inputs and outputs $\widetilde{x}_{ij}$, $\widetilde{y}_{rj}$, $j = 1, \ldots, n$, $i = 1, \ldots, m$, $r = 1, \ldots, s$; the index $p \in \{1, \ldots, n\}$ of the DMU whose fuzzy efficiency is derived.

Set $L^l = \varnothing$, $L^u = \varnothing$ and $\varepsilon = 10^{-12}$.
for $k = \overline{1, q}$ **do**
  Set $\alpha = \alpha_k$.
  Solve Model (6) and derive its optimal value $\theta_k^{\max}$.
  Set $L^r = L^r \cup \{(\theta_k^{\max}, \alpha_k)\}$.
  Solve Model (7) and derive its optimal value $\theta_k^{\min}$.
  Set $L^l = L^l \cup \{(\theta_k^{\min}, \alpha_k)\}$.
end for

Output: The lists L^l and L^r containing the left and right endpoints of $\left[\widetilde{\theta}_p^*\right]_\alpha$ for $\alpha = \alpha_1$, $\alpha = \alpha_2$,..., $\alpha = \alpha_q$ respectively.

Algorithm 4 to compute the relative efficiency of DMU_{j_1} with respect to DMU_{j_2}

Input: The lists $L_{j_1}^l$, $L_{j_1}^r$, $L_{j_2}^l$ and $L_{j_2}^r$ describing the fuzzy efficiencies of the DMUs whose relative fuzzy efficiency is derived.

Set $L^l = \varnothing$, $L^u = \varnothing$, $q = \left|L_{j_1}^l\right|$.
for $k = \overline{1, q}$ **do**
  Set $L^l = L^l \cup \left\{\left(L_{j_1}^l(k) - L_{j_2}^r(k), \alpha_k\right)\right\}$.
  Set $L^r = L^r \cup \left\{\left(L_{j_1}^r(k) - L_{j_2}^l(k), \alpha_k\right)\right\}$.
end for

Output: The lists L^l and L^r containing the left and right endpoints of $\left[\widetilde{\theta}_{j_1}^* - \widetilde{\theta}_{j_2}^*\right]_\alpha$ for $\alpha = \alpha_1$, $\alpha = \alpha_2$,..., $\alpha = \alpha_q$ respectively.

There are various methodologies in the literature that provide conclusions about the efficiency or non-efficiency of a DMU in a FF-DEA mainly by a procedure that dissolves the fuzzyness. Step 4 in Algorithm 2 refers to such conclusions and it can be extended in accordance to any desired criterion. There are also various methodologies to compare the fuzzy efficiencies, but none of the existing methods can be applied to all situations. For our numerical illustration we recall the idea proposed in [22] to derive the relative efficiency of one DMU with respect to another, and to rank them. We adapt it to work with the special approximate form of the membership functions of the fuzzy efficiencies derived following Kao and Liu's methodology [7]; and developed the next three algorithms.

Using Models (6) and (7) for a given DMU_p, Algorithm 3 provides an approximation to the membership function of its fuzzy efficiency derived in accordance to the extension principle described in (4). The value of p is given as input parameter. The outputs of the algorithm are two lists L^l and L^r containing the left and right endpoints of $\left[\widetilde{\theta}_p^*\right]_\alpha$, for certain values of α also specified as inputs.

Algorithm 4 computes the relative efficiency of DMU_{j_1} with respect to the DMU_{j_2} using the lists $L_{j_1}^l$, $L_{j_1}^r$, $L_{j_2}^l$ and $L_{j_2}^r$ that are outputs of Algorithm 3 when it runs for $p = j_1$ and $p = j_2$ respectively.

Algorithm 5 compares any two DMUs using their relative efficiency described by the lists $L_{j_1 j_2}^l$ and $L_{j_1 j_2}^r$ yielded by Algorithm 4. The output parameter f is equal to 1 if $DMU_{j_1} \geq DMU_{j_2}$, otherwise $f = 0$.

4 Computation results

For our experiments we used our new developed Lua programming language libraries [21] that enable modeling and solving linear and nonlinear mathematical optimization problems. Lua is proven to be a robust, powerful, fast, portable and embeddable language; and in the same time simple, small, and free. Our libraries facilitate the use of an open source solver in Lua providing proper modeling

Algorithm 5 that compares the DMU_{j_1} with DMU_{j_2}

Input: The lists $L^l_{j_1 j_2}$ and $L^r_{j_1 j_2}$ that together describe the relative efficiency of DMU_{j_1} with respect to DMU_{j_2}.

Concatenate the list $L^l_{j_1 j_2}$ with the reversed list $L^r_{j_1 j_2}$ obtaining $L = ((l_1, \alpha_1), (l_2, \alpha_2), \ldots, (l_h, \alpha_h))$ with elements re-indexed in accordance to their order of appearance.
if $l_1 \geq 0$ **then** $f = 1$
else
  if $l_h \leq 0$ **then** $f = 0$
  else
    Find the index i such that $(z_i, \alpha_i), (z_{i+1}, \alpha_{i+1}) \in L$ and $z_i < 0 \leq z_{i+1}$.
    Compute $\alpha = \alpha_i - \dfrac{(\alpha_{i+1} - \alpha_i) z_i}{z_{i+1} - z_i}$.
    Insert $(0, \alpha)$ in L between (z_i, α_i) and (z_{i+1}, α_{i+1}) and re-index the elements of L .
    Compute $s^- = \dfrac{1}{2} \sum_{d=1}^{i} (\alpha_{d+1} - \alpha_d)(z_{d+1} - z_d)$.
    Compute $s^+ = \dfrac{1}{2} \sum_{d=i+1}^{h} (\alpha_{d+1} - \alpha_d)(z_{d+1} - z_d)$.
    if $s^+ \geq s^-$ **then** $f = 1$
    else
      $f = 0$.
    end if
  end if
end if

Output: The flag f.

Table 1: Inputs and outputs for five DMUs given by triangular fuzzy numbers

DMU	Inputs 1	Inputs 2	Outputs 1	Outputs 2
A	$(3.5, 4, 4.5)$	$(1.9, 2.1, 2.3)$	$(2.4, 2.6, 2.8)$	$(3.8, 4.1, 4.4)$
B	$(2.9, 2.9, 2.9)$	$(1.4, 1.5, 1.6)$	$(2.2, 2.2, 2.2)$	$(3.3, 3.5, 3.7)$
C	$(4.4, 4.9, 5.4)$	$(2.2, 2.6, 3)$	$(2.7, 3.2, 3.7)$	$(4.3, 5.1, 5.9)$
D	$(3.4, 4.1, 4.8)$	$(2.2, 2.3, 2.4)$	$(2.3, 2.5, 2.9)$	$(5.5, 5.7, 5.9)$
E	$(5.9, 6.1, 7.1)$	$(3.6, 4.1, 4.6)$	$(4.4, 5.1, 5.8)$	$(6.5, 7.4, 8.3)$

and model manipulation capabilities.

4.1 The first numerical example

To illustrate our theoretical statements we first recall a numerical example used in [18] and [13]. For this example five DMUs with two inputs and two outputs were considered. Originally, in [18] both inputs and outputs were designed as Z-numbers.

According to Kang et al.'s method the Z-numbers values were converted into the conventional triangular fuzzy numbers listed in Table 1.

In both papers [18] and [13] the authors used a priori imposed triangular fuzzy number shapes for the desired weights, and derived the fuzzy efficiencies of DMUs with respect to those shapes. In this way they displaced the derived solutions from the realistic solutions that comply to the extension principle.

Our Monte Carlo simulation yielded the empirical solutions graphed in Figures 1, 2, 3, 4 and 5. These figures also contain the numerical results reported in [18] and [13], and the numerical results obtained via mathematical models introduced in [7].

Table 2 shows that DMUs A and C can be considered non-efficient since their efficiency values with maximal amplitude are smaller than 1, while DMUs B, D, and E are all efficient. In what follows,

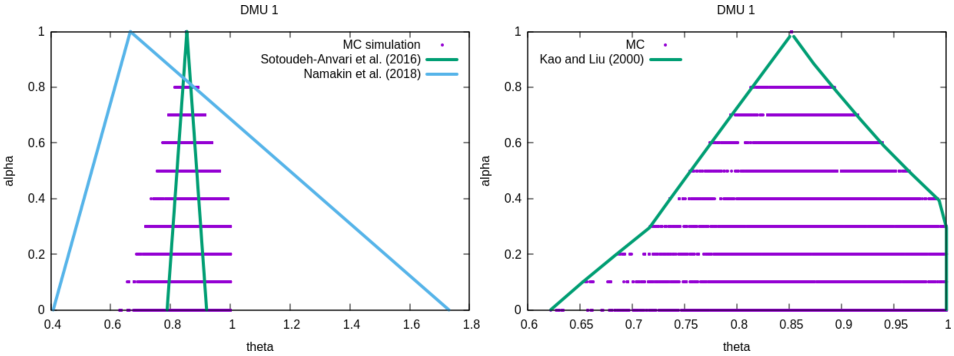

Figure 1: Graphic comparison of the efficiencies obtained for DMU A by our MC simulation, Kao and Liu's solution approach [7], and the approaches presented in [18] and [13] respectively.

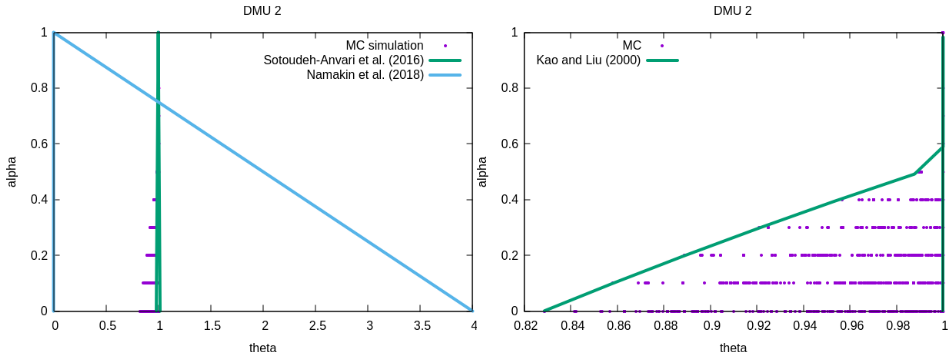

Figure 2: Graphic comparison of the efficiencies obtained for DMU B by our MC simulation, Kao and Liu's solution approach [7], and the approaches presented in [18] and [13] respectively.

aiming to limit the graphic representations, we compare separately DMUs A and C on one side (see Figure 6), and B, D and E on the other side (see Figure 7).

Figure 6 shows that DMU A should be better ranked than DMU C. Figure 7 identifies E as the best ranked DMU; and showes that DMU B should be better ranked than DMU D.

To rank the DMUs with respect to their fuzzy efficiencies we follow Wang et al.'s method [22]. The ranking matrix is presented in Table 3. The comparison between any two DMUs generally establishes a partial order. The results obtained for this particular example show that a complete order can be established between DMUs. The DMUs are listed in their decreasing order in Table 3, thus all upper-diagonal components are equal to 1 in the ranking matrix, and all others are equal to 0.

The final ranking based on extension principle is $E \succeq D \succeq B \succeq A \succeq C$.

Namakin et al. [13] provided a different ranking, namely $B = C = E \succeq D \succeq A$. Note that our approach ranks DMU C on the worst position while Namakin et al.'s [13] considered it as the best

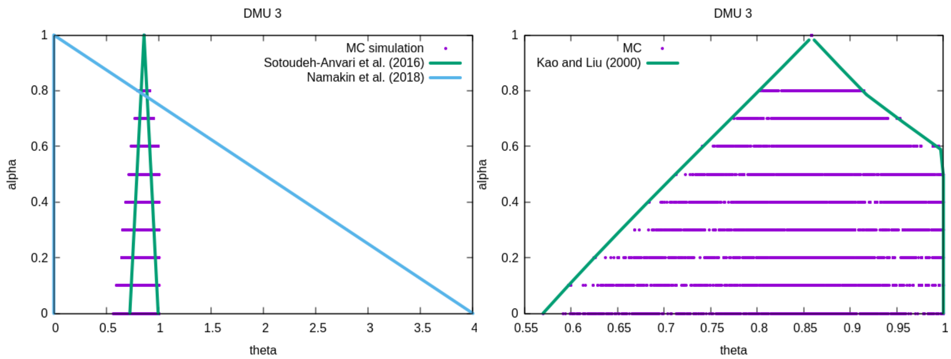

Figure 3: Graphic comparison of the efficiencies obtained for DMU C by our MC simulation, Kao and Liu's solution approach [7], and the approaches presented in [18] and [13] respectively.

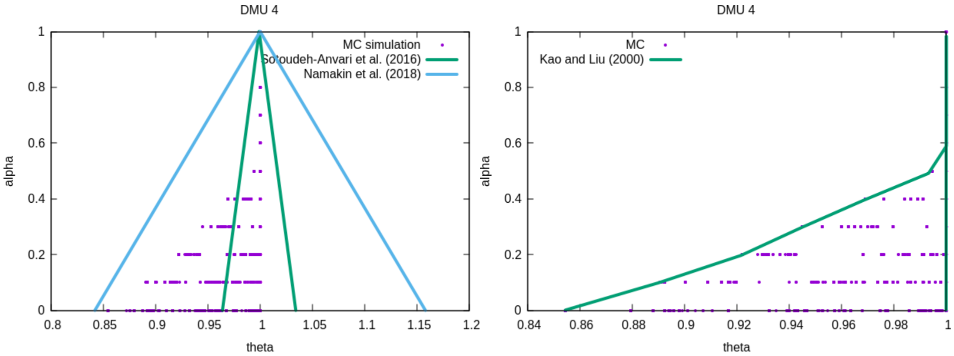

Figure 4: Graphic comparison of the efficiencies obtained for DMU D by our MC simulation, Kao and Liu's solution approach [7], and the approaches presented in [18] and [13] respectively.

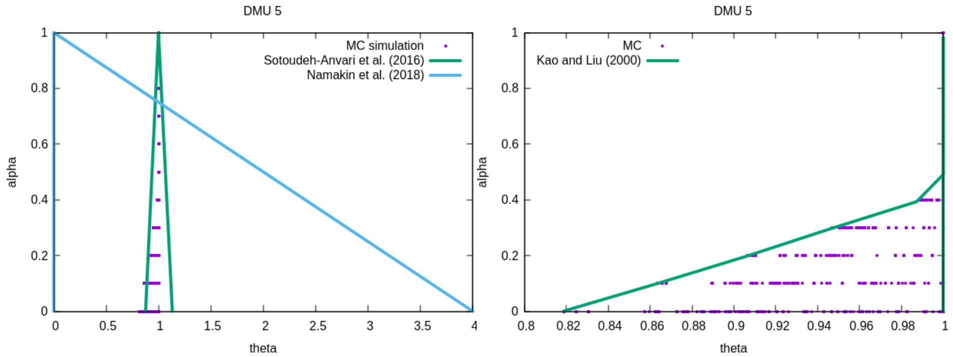

Figure 5: Graphic comparison of the efficiencies obtained for DMU E by our MC simulation, Kao and Liu's solution approach [7], and the approaches presented in [18] and [13] respectively.

ranked DMU, together with DMUs B and E. On the other side, our conclusion matches better the results of Sotoudeh-Anvari et al. [18].

According to Guo and Tanaka [6] DMUs B, D and E are efficient, and DMUs A and C are non-efficient.

4.2 The second numerical example

Our second numerical example is a case-study – the evaluation of seven airline's performances – recalled form [16]. Soltanzadeh and Omrani [16] proposed a fuzzy DEA model and applied it to a real life problem. Their solution approach relied on Kao and Liu's methodology [7], followed the extension principle until the linearization step, and considered crisp weights for their models.

For our experiments we used the crisp model

Table 2: Numerical comparison of our empirical fuzzy number values efficiencies obtained by a Monte Carlo simulation with the results reported in the literature.

DMU	Sotoudeh-Anvari et al. (2016)	Namakin et al. (2018)	Kao and Liu, (2000)	MC simulation, current approach
A	$(0.79, 0.85, 0.92)$	$(0.40, 0.67, 1.70)$	$(0.62, 0.85, 1.00)$	$(0.70, 0.85, 1.00)$
B	$(0.98, 1.00, 1.02)$	$(0.00, 0.00, 4.00)$	$(0.83, 1.00, 1.00)$	$(0.91, 1.00, 1.00)$
C	$(0.73, 0.86, 1.00)$	$(0.00, 0.00, 4.00)$	$(0.57, 0.86, 1.00)$	$(0.63, 0.86, 1.00)$
D	$(0.96, 1.00, 1.03)$	$(0.84, 1.00, 1.16)$	$(0.85, 1.00, 1.00)$	$(0.92, 1.00, 1.00)$
E	$(0.88, 1.00, 1.13)$	$(0.00, 0.00, 4.00)$	$(0.82, 1.00, 1.00)$	$(0.85, 1.00, 1.00)$

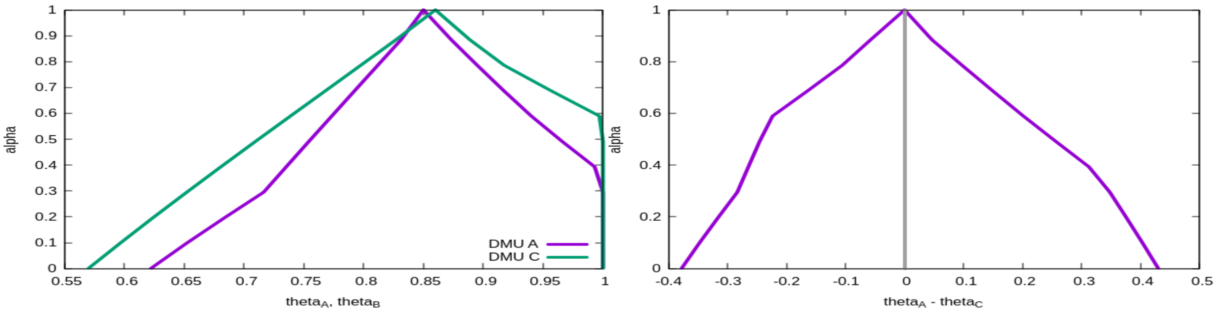

Figure 6: Graphic comparison of the efficiencies obtained with respect to the extension principle for DMUs A and C. It shows that $A \succeq C$.

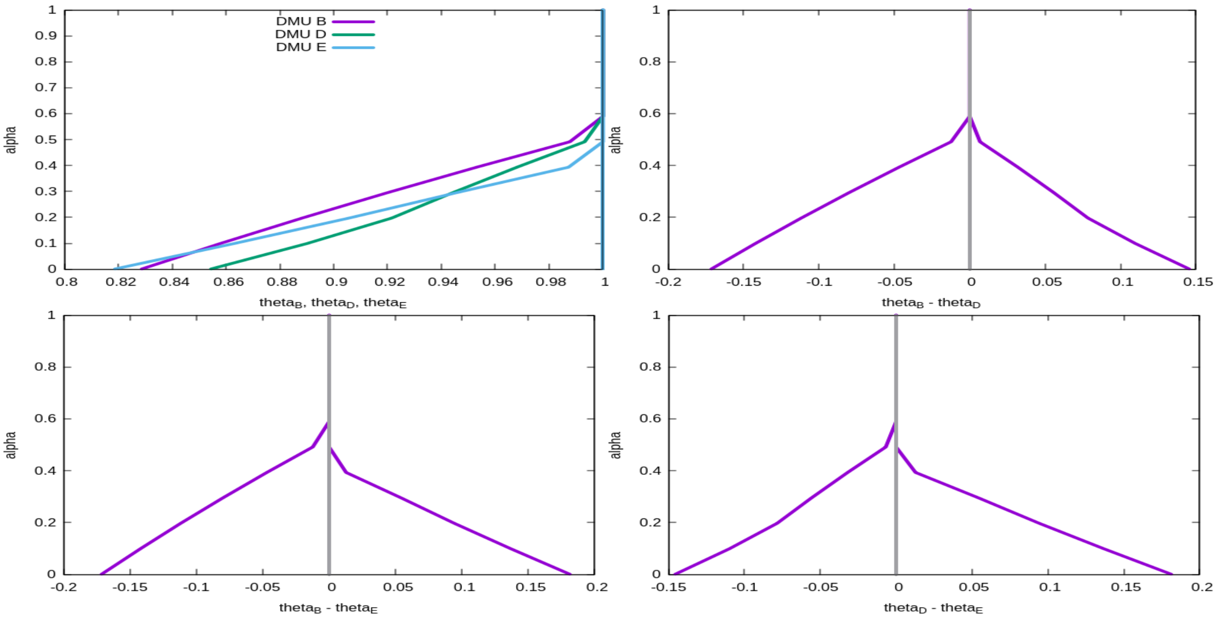

Figure 7: Graphic comparison of the efficiencies obtained with respect to the extension principle for DMUs B, D and E. It shows that $E \succeq D \succeq B$.

$$
\begin{array}{lll}
\max & \displaystyle\sum_{r=1}^{2} u_r \left(\sum_{t=t_1}^{t_3} y_{prt} \right) + f \cdot z_{pt_3} & \\
\text{s.t.} & & \\
& v \left(\displaystyle\sum_{t=t_1}^{t_3} x_{pt} \right) + f \cdot z_{pt_0} = 1, & \\
& \displaystyle\sum_{l=1}^{3} c_{jlt} w_l \leq v \cdot x_{jt}, & j = \overline{1,7}, t = \overline{t_1, t_3}, \\
& \displaystyle\sum_{r=1}^{2} u_r y_{jrt} + f \cdot z_{jt} \leq \sum_{l=1}^{3} w_l c_{jlt} + f \cdot z_{j(t-1)}, & j = \overline{1,7}, t = \overline{t_1, t_3}, \\
& u_r, w_l, v, f \geq \varepsilon, & r = \overline{1,2}, l = \overline{1,3}, \\
& (y_{jrt})_\alpha^L \leq y_{jrt} \leq (y_{jrt})_\alpha^U, & j = \overline{1,7}, r = \overline{1,s}, t = \overline{t_1, t_3}, \\
& (x_{jt})_\alpha^L \leq x_{jt} \leq (x_{jt})_\alpha^U, & j = \overline{1,7}, t = \overline{t_1, t_3}, \\
& (c_{jlt})_\alpha^L \leq c_{jlt} \leq (c_{jlt})_\alpha^U, & j = \overline{1,7}, l = \overline{1,3}, t = \overline{t_1, t_3}, \\
& (\widetilde{z}_{jt})_\alpha^L \leq z_{jt} \leq (\widetilde{z}_{jt})_\alpha^U, & j = \overline{1,7}, t = \overline{t_1, t_3},
\end{array}
$$

recalled form [16], on data given in Table 2 from the same reference, as a particular case of Model (6). Note the additional parameters c and z that were used together with the classic parameters: inputs x, and outputs y. To have the third constraint generally valid, all lower and upper values of

Table 3: The ranking matrix of DMUs obtained with respect to the extension principle

DMU	E	D	B	A	C
E	1	1	1	1	1
D	0	1	1	1	1
B	0	0	1	1	1
A	0	0	0	1	1
C	0	0	0	0	1

the parameters c, z and y for $t = t_0$ were set to 0.

Running our Monte Carlo simulation algorithm with the fuzzy DEA model proposed in [16], we show that our results accurately simulate the results obtained in [16] for the last six DMUs, but also identify an incompatibility to the results of Soltanzadeh and Omrani for the first DMU. The incompatibility is due to the linearization proposed in [16] (that even thought it works for classic CCR models it might not function for complex models without loosing the accuracy). After the identification of the incompatibilities arisen for the first DMU we solved the non-linear models associated with the primal optimization (6) and found that a similar incompatibility appears for the sixth DMU as well. These findings can be seen in Figure 8.

We also use the results obtained by the Monte Carlo simulation for the first DMU to disclose the shapes of the fuzzy weights (see Figure 9) hidden behind the crisp weights used within the solution approach [16].

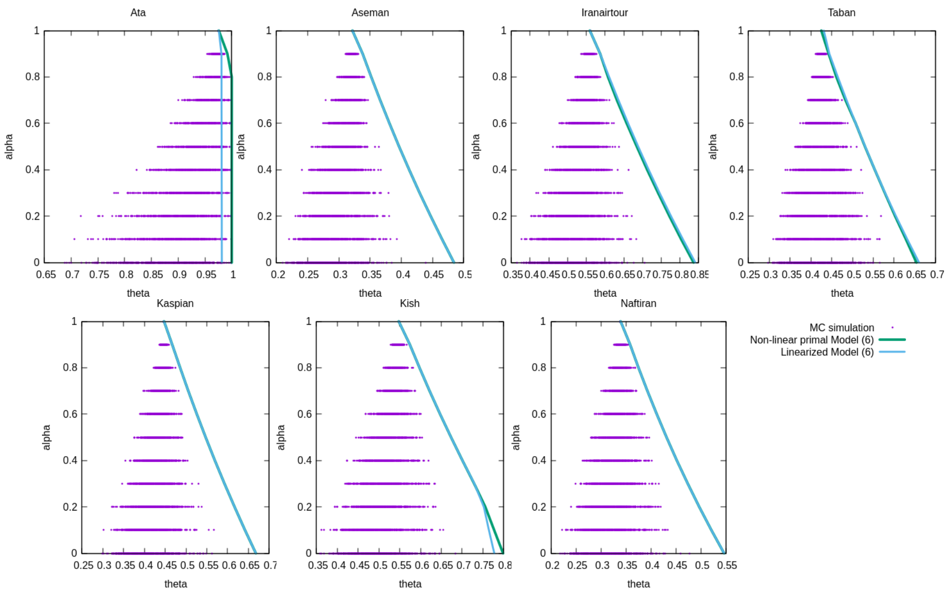

Figure 8: The graphic representation of the fuzzy efficiencies of all DMUs obtained by running our Monte Carlo simulation-based algorithm, and our implementations based on the non-linear Model (6), and on the linearized Model (6) respectively, on data provided in [16], Table 2.

Figure 9 shows the simulated values of the optimal weights of the inputs and outputs for the first DMU that is also first ranked in [16]. Note their shapes that can represent areas under graphs of membership functions of fuzzy numbers.

In this way, we illustrate the fact that the fuzzy data available for a real life problem was used in [16] for fuzzy DEA model but the methodology that was applied there derived solutions that correspond to a full fuzzy DEA model. Figure 9 shows that the graphic representations of the simulated values of the decision variables correspond to membership functions of fuzzy numbers.

5 Conclusions and future works

The main contribution of our paper is twofold: (i) we provided a simple, general and powerful Monte Carlo simulation-based algorithm that can be used to check whether a solution approach to

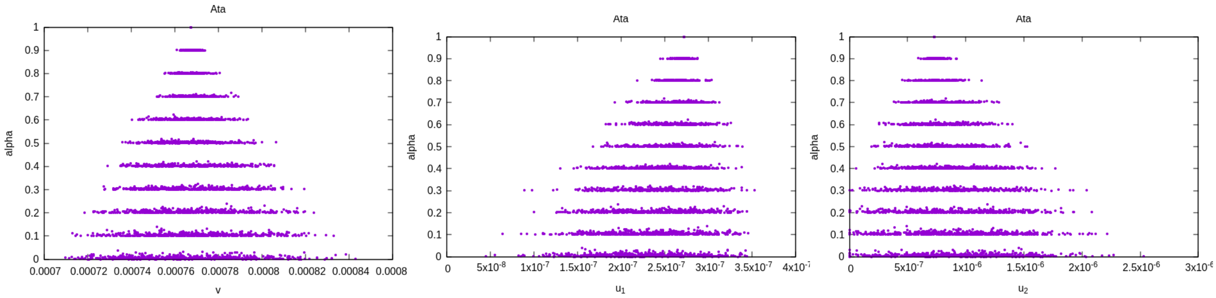

Figure 9: The graphic representation of the fuzzy weights $\widetilde{u}_1$ and $\widetilde{u}_2$ of the outputs, and $\widetilde{v}$ of the input obtained by the Monte Carlo simulation for the first DMU.

full fuzzy DEA derives solutions with respect to the extension principle or not; and (ii) we showed that a methodology based on the extension principle and initially formulated for fuzzy DEA can be successfully used to derive solutions to full fuzzy DEA, since the crisp weights used within the mathematical models, in fact, obscurely shape the fuzzy weights of a corresponding full fuzzy model.

To illustrate our theoretical statements we used two numerical examples from the recent literature. The first example was the full fuzzy variant of the classic DEA model. In the literature it was solved using methodologies that do not comply to the extension principle, thus offering us the opportunity to emphasize the misleading consequences of such approaches.

The second example dealt with a more complex extended DEA model with fuzzy coefficients, and the methodology found in the literature followed the extension principle in deriving crisp non-linear mathematical models. However the liniarization used to simplify the solution approach led to a loss of accuracy that was identified by our Monte Carlo simulation algorithm.

Summarizing, our main goals were to provide quadratic optimization models able to derive exact solutions to small-size problems; to propose linear models whose solutions can disclose relevant information about the efficiencies of DMUs to large-scale problems, and in full accordance to the extension principle; and, the most general, to rehabilitate the use of the extension principle within fuzzy DEA.

Our further research will include a wider examination of DEA models under fuzzy environment. A special attention will be payed to identifying the reasons that make certain DEA methodologies to derive results quite distant from the more realistic results that can be obtained by following the extension principle.

Author contributions

The authors contributed equally to this work.

Conflict of interest

The authors declare that they have no conflict of interest.

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